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SCHEUBEL AS AN ALGEBRAIST

BEING A STUDY OF ALGEBRA IN THE MIDDLE OF THE SIXTEENTH CENTURY, TOGETHER WITH A TRANSLATION OF AND A COMMENTARY UPON AN UNPUBLISHED MANUSCRIPT OF SCHEUBEL'S NOW IN THE LIBRARY OF COLUMBIA UNIVERSITY

BY

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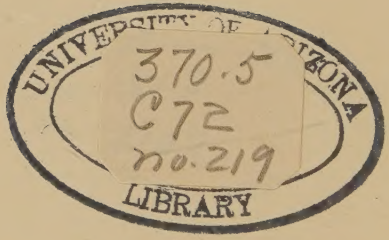

PREFACE AND ACKNOWLEDGMENT

We are living in an age when the primitive phases of all arts and sciences are arousing interest and receiving much consideration. First beginnings are always fascinating and illuminating. As early as 1869 Hankel considered the development of mathematics as an integral part of the history of civilization. The sixteenth century, with its far-reaching changes in all lines of human activity, is a period of great interest and fascination for the historian. From the standpoint of the history of science, which stresses the accumulation of new material from the various fields of scientific works, no justification is needed for the publication of an unknown mathematical work, an unpublished algebra of about the year 1550. Through a study of algebra in the middle part of the sixteenth century we hope to produce a more just estimate of the services rendered by mathematics.

The author is deeply appreciative of the services of Professor David Eugene Smith of Teachers College, Columbia University, whose suggestion, inspiration, and guidance have made possible this research.

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M. S. D.

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SCHEUBEL AS AN ALGEBRAIST

I

INTRODUCTION

Development in mathematics, as in art and literature, is not a perfectly logical process, but may be best studied in advancing and receding currents, altogether constantly advancing.

The algebraic side of mathematics is not, as is sometimes assumed, an artificial experiment of human ingenuity. Rather it expresses the intrinsic interest of the human mind in the numerical side of the universe. Since it is dependent upon the development of reasoning, various contributing influences enter into the process of an evolution which satisfies man's intellectual and practical needs.

Before the sixteenth century, printing was not sufficiently established to permit free dissemination of thought, but during this period superstition began to give way to scientific inquiry. Each previous century had contributed but few names to represent the growth of mathematics. At this time, through the printed page, many mathematicians entered the world of scientific thought. The conditions of the field of mathematics were such as to mark out the course of progress in the domain of analysis, to include a better symbolism, and to permit of the solution of the cubic and biquadratic equations and of numerical equations of higher degrees. In fact, in the sixteenth century mathematics experienced a more extensive development than it had known since the School of Alexandria produced its numerous scholars.

If we should attempt to give a definition of algebra, a term so loosely used as to make precise definition difficult, we should have to consider a meaning which would be applicable to the different stages of its development. Certain problems of elementary mathematics have from time to time furnished the occasion for the development of more advanced subjects, and as these new subjects are acquired, they have helped to illuminate the elementary parts.

The essential concepts of algebra can best be enumerated when

they are considered historically. Nesselmann[1] represents the three periods of the development of algebra by dividing it into three general types: (1) the rhetorical, in which there are no symbols and the words and sentences are written out in full; (2) the syncopated, in which certain abbreviated symbols condense the work for constantly recurring quantities and operations; (3) the symbolic, in which a system of notation and a language of symbols permit of further abbreviation. The first period, pre-Grecian, includes the classical period; the second contains the late Greek and mediæval; the third is modern.

[1] H. G. F. Nesselmann, *Die Algebra der Griechen*, Berlin, 1842, p. 302; hereafter referred to as Nesselmann, *Algebra*.

II

GENERAL STATUS OF ALGEBRA BEFORE THE SIXTEENTH CENTURY

1. WHAT EGYPT HAD DONE

Algebraic equations were found, usually in rhetorical form, in the oldest of the extensive mathematical manuscripts, the *Ahmes Papyrus* (c. 1550).[1] In the so-called *hau* computation (*hau* being a hieroglyphic combination which means "mass" or "heap") such a linear equation with one unknown quantity is solved by the "rule of false position,"[2] a systematic guessing at an answer, finding the error, and then modifying the guess accordingly, as: "Mass, its whole, its fifth, it makes 21."[3] It contains problems involving elementary series and arithmetical and geometrical progression,[4] as well as problems which seem to imply a rudimentary knowledge of proportion. No good algebraic symbolism is used in the treatment of problems and there is no evidence that algebra existed as a science.[5]

The *Kahun Papyrus* (c. eighteenth century B.C.),[6] which contains the earliest quadratic equations of which we have a record, bears witness that algebra made no further progress among the Egyptians.

2. WHAT CHINA HAD DONE

Because we are too uncertain regarding the early literature of the Chinese to be positive as to the dates and authenticity of their classics, it is difficult to say when algebra began as a science in China. In Liu Hui's commentary (c. 250) on the *Nine Sections* we find a crude notation for determinants and rules for solving rhetorical quadratic equations.[7] Sun-tzï's work (c. first century)

[1] T. E. Peet, *The Rhind Mathematical Papyrus*, London, 1923, hereafter referred to as Peet, *Ahmes*.
[2] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 4 vols., Leipzig, 1880–1908, vol. I (3), pp. 78–9, 95; hereafter referred to as Cantor, *Geschichte*.
[3] Peet, *Ahmes*, p. 62, No. 27.
[4] Peet, *Ahmes*, p. 78, No. 40.
[5] Peet, *Ahmes*, p. 60.
[6] Cantor, *Geschichte* I (3), pp. 78–9, 95.
[7] Y. Mikami, *The Development of Mathematics in China and Japan*, Leipzig, 1913, pp. 17–19, 23; hereafter referred to as Mikami, *China*.

contains questions involving indeterminate equations whose solution by analysis gives only one result.[1]

The Chinese algebraists, chiefly interested in applied problems, did nothing worthy of note with the general cubic equation. The first numerical cubic appeared in a work by Wang Hs'iao-t'ung (c. 625).[2]

China's greatest contribution to algebra was certain ingenious devices for treating numerical higher equations. Such scholars as Ch'in Kiu-shao (c. 1250), Li Yeh (c. 1250), and Chu-shï-kié (c. 1300)[3] perfected an ancient method which resembled the one later developed by W. G. Horner (1819).

But when Jesuit missionaries came during the sixteenth century and introduced Western science, China lost her native algebra and she never fully regained it.

3. WHAT THE GREEKS OF THE PRE-CHRISTIAN ERA HAD DONE

Algebra in the modern sense can hardly be said to have existed in the golden age of Greek mathematics, but the ancient *arithmetica*, or theory of numbers, commonly represented by lines and rectangles, the geometric treatment of formulas, square root,[4] cube root,[5] and proportion (Euclid's *Elements*, Bk. V), the duplication of the cube by the aid of geometrical curves,[6] as well as the first systematic solution of quadratic equations by means of the completion of geometric squares, may properly be considered as a phase of algebra.[7] The first scientific expression of the meaning of irrationals, that is now extant, is found in Euclid's definition of proportion (Euclid's *Elements*, V).

The Greeks greatly simplified the treatment of the series already known, they began a development of certain higher types, and they first saw the possibilities of infinite series.[8]

After Euclid, the analytic method, with its employment of a fairly good symbolism, gradually replaced the geometric, and

[1] Mikami, *China*, pp. 35, 36.
[2] Mikami, *China*, p. 54.
[3] Mikami, *China*, pp. 63, 79, 89.
[4] Cantor, *Geschichte*, I (2), pp. 442, 460–461.
[5] J. Tropfke, *Geschichte der Elementar Mathematik in systematischer Darstellung*, 7 vols., 2nd ed., 1922–, III, pp. 1–19; hereafter referred to as Tropfke, *Geschichte*.
[6] T. L. Heath, *Diophantus of Alexandria*, 2nd ed., Cambridge, 1910, p. 63; hereafter referred to as Heath, *Diophantus*. "The geometrical equivalent of the solution of a quadratic was assumed as early as the fifth century B. C. by Hippocrates of Chios in his *Quadrature of lunes*. The complete geometric solution is in Euclid VI, 127–129."
[7] Heath, *Diophantus*, pp. 50, 65, 179, 181, 228–229.
[8] David Eugene Smith, *Mathematics*, Boston, 1923, p. 83; hereafter referred to as Smith, *Mathematics*.

Heron (c. 50?) may have used it in his solution of quadratics involving imaginary roots.[1]

4. WHAT DIOPHANTUS HAD DONE

Diophantus of Alexandria (c. 275), whose work was known to the Arabs but was not really appreciated until it was discovered in Europe in the sixteenth century,[2] stands out in the history of science as one of the great unexplained geniuses. Because he replaces the old geometric method by the algebraic method,[3] because he first uses an algebraic symbolism worthy of the name and as good as any that had appeared before the sixteenth century, and because his *Arithmetica* is the first treatise almost entirely devoted to algebra,[4] he is often considered as the father of the science.[5] In Book I he is very skillful in handling determinate equations of the first and second degrees, in which he uses clever devices to express his problems in terms of only one unknown[6] and which he was able to simplify by transposition.[7] His simultaneous quadratics relate only to special cases.

Since he did more than any other Greek writer with indeterminate equations of the second degree, his favorite subject, equations of this type are called Diophantine equations.[8]

5. WHAT INDIA HAD DONE

There are only four Hindu algebraists whose names are particularly noteworthy. In the work of Āryabhata (c. 475–550), as given by Rodet,[9] there is a rule for summing an arithmetical series and he shows a knowledge of quadratic equations and of indeterminate linear equations.[10] Brahmagupta (c. 628) gives rules for negative numbers in his chapter on quadratic equations[11] and a

[1] Cantor, *Geschichte*, I (3), pp. 374, 377, 405.

[2] Heath, *Diophantus*, pp. 17, 22–25, "Xylander (Wilhelm Holzmann) gave the first Latin edition of his works, Basel, 1575."

[3] "Although most algebraists to about 1700 A. D. employed geometric figures more or less, he nowhere appeals to them," H. Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, Leipzig, 1874, p. 162; hereafter referred to as Hankel, *Geschichte*.

[4] Cantor, *Geschichte*, I (3), p. 377.

[5] Hankel, *Geschichte*, p. 158.

[6] Heath, *Diophantus*, pp. 51, 58.

[7] *Ibid.*, p. 131.

[8] D. E. Smith, *History of Mathematics*, New York, 1923, 2 vols., II, p. 382; hereafter referred to as Smith, *History*.

[9] "Leçons de Calcul d'Āryabhata," *Journal Asiatique*, XIII (7), pp. 393–434.

[10] Cantor, *Geschichte*, I (3), p. 575, "Āryabhata made some contributions to algebra, but did not carry the subject materially farther than the Greeks."
Smith, *Mathematics*, p. 130, "The Hindus added nothing worthy of note to the stock of algebraic knowledge except in the way of a symbolism which no later writers adopted and in the way of numerous interesting problems."

[11] H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrît of Brahmegupta and Bhâscara*, London, 1817, pp. 347, 374; hereafter referred to as Colebrooke, *Algebra*; *Geschichte*, I (3), p. 625.

correct rule for obtaining the positive root of a quadratic by completing the square. His fanciful problems include simultaneous equations of the first degree. Mahāvīra (c. 850), in his *Gaṇita-Sāra-Sangraha*,[1] includes three types of complete quadratic equations and work on series and radicals. Bhāskara's *Bija Gaṇita* (c. 1150)[2] includes both algebraic and geometric solutions in indeterminate problems which involve two unknowns.[3] His treatment of simple and quadratic equations is more clear than that of any other Hindu writer and includes problems relating to geometric figures and the usual poetic types.

6. WHAT THE MOHAMMEDANS HAD DONE

The Mohammedans received algebra chiefly from the Greeks but they revised and systematized the material to some extent. Because they preserved in their translations many of the important works of the Hindus and Greeks, Europe is chiefly indebted to them as transmitters rather than as creators.

Al-Khowârizmî (c. 825), the leader of the Bagdad school, wrote the first general work of any importance upon algebra and to the science he gave its name.[4] He discusses the solution of simple and quadratic equations in a scientific manner, distinguishing six different classes, and he uses the equivalent of to-day's formula for finding two roots when both are positive.[5]

Noteworthy among the Mohammedans who showed a one-to-one correspondence between algebra and geometry for developing rules and for the solution of the quadratic were al-Khowârizmî, Abû-Kamil (c. 900), al-Karkhî[6] (c. 1020),[7] and Omar Khayyam (c. 1100).[8]

7. WHAT FIBONACCI HAD DONE

Leonardo Fibonacci (c. 1202), the outstanding scientific genius of the thirteenth century, in his *Liber Abaci*[9] uses geometry to

[1] Translated by M. Rañgācārya, Madras, 1912.
[2] Colebrooke, *Algebra*, pp. 129–276.
[3] Colebrooke, *Algebra*, p. 270.
[4] *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Leipzig, 1877, vol. II, pp. 1–124. "Die deutsche Coss," P. Treutlein, p. 5, "Im Anfange des 9. Jahrhunderts verfasste der aus Kharizm . . . mit dem Zunamen Alkharazmi . . . ein, populäres Werk über Algebra. . . '*Al gebr w'al mukabalah*' . . . *al gebr* (von *gabar=restaurare*. . .) . . . und *al mukabala* (von *kabala=opponere, comparare* . . .) . . .;" hereafter referred to as Treutlein, *Die Coss*.
[5] F. Rosen, *The Algebra of Mohammed ben Musa*, London, 1831, p. 11; hereafter referred to as Rosen, *al-Khowârizmî*.
[6] Rosen, *al-Khowârizmî*, p. 13.
[7] F. Woepcke, *Extrait du Fakhrî . . Alkarchî*, Paris, 1853, Preface, p. v. See Smith, *History*, II, Chap. VI, for an explanation of *Fakhrî*.
[8] F. Woepcke, *L'Algèbre d'Omar Alkhayyâmî*, Paris, 1851, p. 13.
[9] B. Boncompagni, *Scritti di Leonardo Pisano*, 2 vols., Rome, 1857–1862; I, pp. 401–459; hereafter referred to as Boncompagni, *Scritti*.

demonstrate algebraic formulæ and simple, quadratic and indeterminate equations. His *Liber Quadratorum*,[1] solutions to questions involving quadratic and cubic equations asked him by Frederick II, portrays an unusual treatment of the theory of numbers, and it is a proof of his ability as a remarkable mathematician. He is very skillful in finding results for such problems, the greater part of which are original, as: "Volo inuenire tres numeros, qui insimul aggregati cum quadrato primi numeri faciant quadratum numerum."[2] He was a standard authority for centuries and from him writers drew their inspiration.[3]

8. HOW PACIOLI HAD LEFT THE SUBJECT

Luca Pacioli in his *Sūma* (1494),[4] the first serious consideration of algebra in the Renaissance period, a remarkable but not an original compilation,[5] presents in his own language all that was known of algebra in his time and draws the attention of the greatest masters to this field. He divides quadratic equations into the three historical classes, such as $x^2 \pm mx = \pm n$, given by Fibonacci, and he demonstrates all these cases geometrically,[6] giving only positive roots and recognizing the impossibility of $\sqrt{-a}$. He solves his problems by the "rule of false,"[7] and he gives the first consideration to be found in printed books[8] to problems involving permutations such as working out the number of persons at a table when $n=1, 2, 3, \ldots 11$.

[1] Boncompagni, *Scritti*, II, pp. 227–283.
[2] Boncompagni, *Scritti*, pp. 279–283.
[3] P. Cossali, *Origine, trasporto in Italia, primi progressi in esse dell' Algebra*, 2 vols., Parma, 1797–1799; I, p. 6.
[4] See D. E. Smith, *Rara Arithmetica*, Boston, 1918, pp. 54, 55, for the complete title; hereafter referred to as Smith, *Rara*.
[5] "He borrowed freely from Ptolemy, Fibonacci and Jordanus, but he did not mention his indebtedness." Smith, *History*, I, p. 252.
[6] *Sūma* (1494), fol. 145, *v*; fol. 146, *v* et passim.
[7] *Sūma* (1494), fol. 145, *r*.
[8] *Sūma* (1494), fol. 43, *v*.

III

WHY THE MATHEMATICIANS OF THE SIXTEENTH CENTURY WERE CHIEFLY INTERESTED IN ALGEBRA

In order to comprehend the cause of the remarkable advance of algebra in the sixteenth century it is necessary to consider briefly the different influences which have been more or less potent at the different contributory stages. In the early traces of algebra we find a geometrical treatment used almost entirely. The natural effect of Euclid's work was to make people conscious that elementary geometry had reached perfection and that the successive generations must progress in another field of mathematics.[1]

Algebra gradually approached the scientific stage in the three centuries that followed the death of Pythagoras (501 B.C.), and with Diophantus (c. 275) the analytic treatment came into use. Algebra next flourished almost exclusively in the East. Then at the School of Bagdad the Mohammedans became the custodians of the Indian and Greek sciences and their leading writer, al-Khowârizmî (c. 825), gave the name to algebra and distinguished the six historical classes of simple and quadratic equations.

The twelfth century was to Christian Europe a period of translations, the thirteenth century represented a period of awakening of the world from an intellectual stupor, a revival and development of algebra in China, in India and in Europe, with Fibonacci (c. 1202) as the outstanding mediæval writer. The establishing of the universities in the thirteenth, fourteenth, and fifteenth centuries has been the most potent influence in the development of the world's mathematical knowledge.

The science of algebra, almost stationary in the period from Fibonacci to Pacioli, received its earliest improvements in Italy. There through a natural outgrowth of circumstances, influenced by the invention of printing, a true renaissance in mathematics, which culminated in one of the leading developments of algebra in the sixteenth century, was effected. At the close of the fifteenth

[1]Smith, *History*, I, p. 107.

and during the sixteenth century the algebraic art gradually approached a modern development. Germany, influenced by such writers as Stifel and Scheubel, contemporary with Cardan and Tartaglia, moved toward the front in the mathematical progress of Europe, but with Italy always in the lead.[1]

Thus we see that the conditions in the field have been such as to mark out a course of progress in algebra to include a better symbolism, and to permit of the solution of the cubic and biquadratic equations and of numerical equations of higher degrees. In fact, in the sixteenth century we find that mathematics, with its chief interest in algebra, experienced a more extensive development than it had known since the school of Alexandria produced its numerous scholars.

[1] Smith, *History*, I, p. 416.

IV

WHAT THE SIXTEENTH CENTURY CONTRIBUTED BEFORE SCHEUBEL'S TIME

1. METHODS OF TREATING THE EQUATION AS TYPIFIED IN THE DEVELOPMENT OF THE CUBIC EQUATION

The much sought for solution of the cubic equation, which had been treated by the Greeks and Arabs only for special cases, is the chief contribution of Italy to the great advancement made in mathematics during the sixteenth century, a period which gave rise to modern elementary algebra.

Cardan, in his *Ars Magna* (1545),[1] the first great Latin treatise devoted solely to algebra, includes the solution of the cubic equation, which he had obtained under the pledge of secrecy from Tartaglia, and of the quartic equation which Ferrari had solved.[2]

Cardan is famous for his originality in introducing symmetric functions with a statement that the sum of the roots is minus the coefficient of x^2.[3] He recognizes negative roots but he calls them *numeri fictae* and attaches to them no independent significance.[4]

2. BROADENING THE SCOPE OF ALGEBRA

(*a*) *Rudolff.*

Christoff Rudolff's *Coss* (1525) made the science of algebra known in Germany,[5] and popularized its development in a manner similar to the effect of Pacioli's *Sūma* in Italy. Stifel's famous edition of the *Coss* (1553–1554) is an improvement on Rudolff's presentation and contains the first modern use of the symbol for square root, $\sqrt{\ }$, Stifel's varied forms:

$\mathcal{z}$/ for $\sqrt{\ }$, $\mathcal{z}$/ for $\sqrt[3]{\ }$, $\mathcal{z}$/ for $\sqrt[4]{\ }$, and so on.

[1] See Smith, *Rara*, pp. 338–9, for the complete title. Cardan included his general solution of the cubic in the last chapter of the *Ars Magna*.
[2] *Ars Magna*, fol. 73, *v* and fol. 74, *r*.
[3] *Ars Magna*, fol. 39, *v*.
[4] *Ars Magna*, Cap. XVIII, fol. 39,
[5] Cantor, II (2), p. 398.

a modern treatment of the rules for the signs + and − involved in the four fundamental operations.[1]

Rudolff was the first representative of this period to use equations with more than one unknown, but he treats of them rather incidentally. He gives rules for the solution of equations of the first and second degrees, inclusive of three types of irrational quadratics, and is the sole representative of the German writers who made a noteworthy attempt to solve the cubic equation.[2] This attempt, however, was a failure.

(*b*) *Stifel.*

Michael Stifel's *Arithmetica Integra* (1544), which was almost entirely algebra, raises the standard of the science just as the works of Cardan and Tartaglia had done in Italy.[3] Following Cardan's method he is able to give a brief account of the theory of equations of the third degree.[4] He even goes a step beyond his predecessors in a solution by approximation, or "regula falsi," of equations of a higher degree. He classifies irrationals[5] and, that he may be able to develop the extraction of roots to a considerable extent, he works out a table of binomial coefficients as far as $(a + b)$. In his geometrical progression he gives the same summation formula,

$$\frac{s - a(ra^{n-1} - a)}{ar - a},$$

which Tartaglia[6] had used, and compares it with arithmetical progression. Stifel also lays the foundation of a concise system of notation when he illustrates positive, negative, and fractional exponents.[7]

3. RECOGNITION OF MODERN SYMBOLISM

"Our present notation has arisen by almost insensible degrees as convenience suggested different marks of abbreviation to dif-

[1] Cantor, *Geschichte*, II (2), p. 444.
[2] Rudolff's *Coss* (1553), fol. 477, *r*.
[3] Smith, *Rara*, pp. 223–226.
[4] *Stifel, Arithmetica Integra*, Nürnberg, 1544, Chap. XIII (Lib. III), fol. 251, *v*; seq. He makes the following reference to Cardan: "Epilogiuice repetit mentione faciens perfectionis regulae Algebrae, cũ exemplis quibusdam Hieronymi Cardani ad hanc rem pertinentibus;" hereafter referred to as Stifel, *Arithmetica Integra*.
[5] Stifel, *Arithmetica Integra* (1544), Bk. II, Chap. I–IX, pp. 103–127; binom. and resid. p. 127 seq. (Euclid X).
[6] Nicolo Tartaglia, *General Trattato*, 1556, II, fol. 6, *r*.
[7] Stifel, *Arithmetica Integra*, p. 249, *v*.

ferent authors; and that perfect symbolic language which addresses itself solely to the eye, and enables us to take in at a glance the most complicated relations of quantity, is the result of a large series of small improvements."[1]

Algebra was encumbered with many difficulties because of the need of a good symbolism. The terms *plus* and *minus* had been used by Fibonacci as early as 1202, and the letters p and m[2] by Chuquet in 1484, and by Pacioli in 1494. In 1489 even, the signs + and − were used as expressions of excess and deficiency in commercial transactions by Widman, but the first use of these signs in algebraic expressions was made by the Dutch mathematician, Vander Hoecke (1514), in the "rule of false," and as symbols of operation.[3] Henricus Grammateus (d. 1525) made the same applications but they were popularized and brought into general use by Stifel. These Teutonic forms were early adopted by the English and we find Recorde (c. 1542) saying, "+ betokeneth more, − betokeneth lesse."[4]

The symbol for square root, $\surd$, was first used by Rudolff (1525)[5] and then by Stifel[6] (1554), Pacioli[7] (1494) had used the symbol R$\frac{x}{}$ and Ramus (c. 1550) had used L (latus). The following examples show the development of the root universal:—Chuquet (1484), R$\frac{x}{}^2$. 14 p. R$\frac{x}{}^2$. 180, $\sqrt{14+\sqrt{180}}$; Pacioli (1494), R$\frac{x}{}$ $\surd$ 40 p. R$\frac{x}{}$ 320, $\sqrt{40+\sqrt{320}}$; Stifel (1534), $\surd 3 \cdot \surd 3$ $128+\surd 3\,92$, $\sqrt{\surd 128+\surd 92}$; Cardan (1545), R$\frac{x}{}$V: cu.R$\frac{x}{}$ 108m: 110, $\sqrt{\sqrt[3]{108}-110}$; Rudolph (1553), $\surd$des collects[8] $12+\surd 40$, $\sqrt{12+\surd 40}$.

The first printed equality sign, appeared in Robert Recorde's *Whetstone of witte* (1557) in which he says, "I will sette as I doe often in worke use, a paire of paraleles, thus: ═══ bicause noe. 2 thynges, can be moare equalle."[9]

[1] J. F. W. Herschel, "Mathematics" in the *Edinburgh Encyclopedia*.
[2] Tropfke, *Geschichte*, II (2), pp. 131–133.
[3] Smith, *Rara*, p. 183; Vander Hoecke, *Arithmetica* (1537 ed.), fol. 64, *v*.
[4] Robert Recorde, *Ground of Artes* (1558 ed.), fol. 6; *Whetstone of witte* (1557), fol. S2, *v*.
[5] Rudolff, *Die Coss*, fol. 139, *v*.
[6] Stifel, *Arithmetica Integra*, fol. 158 *v*.
[7] Colebrooke, *Algebra*, p. XIII.
[8] Compare with Scheubel's (c. 1550), "Radix Collecti," p. 19 of our MS. "Ist der Radikand, über den sich die Wurzel erstrecken sollte, ein zusammengesetzter, so musz Rudolff das Wort Collect einschalten: $\surd$ des collects, $12+\surd 40$, Rudolff's *Coss* (1525), Buch I, Kap. VII," Tropfke, *Geschichte*, I (1), p. 219.
[9] Recorde, *Whetstone of witte*, 1557, fol. Ff, *v*.

4. CRYSTALLIZING OF THE NAME FOR ALGEBRA

The first appearance of the word, algebra, was in a work by al-Khowârizmî (c. 825),[1] al-jabr w'almuqâbalah ("the science of reduction and cancellation"), a title which appeared in thirteenth century Latin as *Ludus algebrae almucgrabalaeque*,[2] in sixteenth century English as algeber[3] and in modern English as algebra[4]. Among the late writers of Latin the name was attributed to Geber, an Arab mathematician, whom they considered as the inventor of the science of algebra.[5]

Among the names given to algebra, which have originated from the terms applied to the unknown quantity, Latin writers have used *Ars rei et census* and *Ars rei*,[6] because the unknown and its square were designated, respectively, by the terms *res* and *census*, Italian writers *Regola o l'arte della cosa*, German writers *Die Coss* and English writers *Cossike arte*. In Italy, during the fifteenth and sixteenth centuries, algebra was called the greater art, as opposed to ordinary arithmetic, the lesser art. But after many vicissitudes of changes in name, algebra is the only one remaining in use.[7]

5. TYPES OF PROBLEMS

Because of the relation of mathematics to the daily life of the people, types of problems illustrate the history of civilization at its different stages of development. The first type of problem in applied mathematics, the number puzzle, may at times bear the stamp of "fictitious and pleasing puzzles," but these were first used in the *Ahmes Papyrus* (c. 1550) as practical applications of the necessities of life. In the same way the problems of the pipes filling[8] a cistern, an evolution of algebraic methods through arithmetic, originated in the public fountain of the Romans. They

[1] This is found on pages 71–122 of our MS. by Johann Scheubel and was translated by Louis Karpinski as *Robert of Chester's Translation of al-Khowârizmî*, New York, 1915. Compare with the reference to Rosen, *al-Khowârizmî*, p. 8.

[2] Boncompagni, *Scritti*, I, p. 405.

[3] Recorde, *Whetstone of witte*, fol. Ee4, *v*. "The rule of equation, commonly called Algebers rule, after the name of the inventoure." "Im sechzehnten Jahrhundert tritt das zweite Wort Almucabala nur noch selten auf; Rudolff, Stifel, Scheubel, Cardano, Gemma Frisius, u. a. haben nur Algebra," Tropfke, *Geschichte*, I (1), p. 153.

[4] Al-Khowârizmî's title had been adopted by European scholars as shown by Robert of Chester's translation (c. 1140), *Liber Algebrae et Almucabola*. John Wallis, *A Treatise of Algebra*, London, 1685, calls it "the art of restitution and comparison, or resolution and equation;" hereafter referred to as Wallis, *Algebra*.

[5] Cantor, *Geschichte*, II (2), p. 165.

[6] Treutlein, *Die Coss*, p. 9.

[7] J. E. Montucla, *Histoire des Mathématiques*, 2nd ed., 4 vols., Paris, 1799–1802. I. pp. 381–4.

[8] D. E. Smith, "On the Origin of Certain Typical Problems," *American Mathematical Monthly*, Vol. XXIV, No. 2, Feb., 1917, pp. 64–71; hereafter referred to as Smith, *Problems*

first took definite form with Heron (c. 50?), they were used by Diophantus (c. 275), by Bhāskara (*Lilāvati*, c. 1150), and they next appeared in the sixteenth century with such writers as Tonstall (1522), Gemma Frisius (1540) and Robert Recorde (c. 1540).[1]

The problem of the Turks and Christians, which has had a history of over 1900 years, was mentioned by Chuquet and popularized by Cardan and Ramus.[2] The testament problem, next in antiquity, found favor with Rudolff (1526), with Cardan (1539) and with Recorde (1546).[3] The partnership problem and its variant, the pasturage problem, were considered very important in the sixteenth and seventeenth centuries and are found in the works of Scheubel, Recorde, and Clavius (1583). The fact that Bachet[4] and Ball[5] have included in their texts problems taken from the writings of Pacioli, Tartaglia, Cardan and other writers of the sixteenth century and that these same problems are found in our modern texts is evidence of that period's influence.

[1] *Ground of Artes* (1558 ed.), fol. A7, *v.*
[2] Smith, *Problems*, p. 68.
[3] Smith, *Problems*, pp. 68, 69.
[4] Claude Gaspar Bachet, *Problemes plaisans et delectables*, Lyons, 1612.
[5] W. W. R. Ball, *Mathematical Recreations and Essays*, 3rd ed., London, 1896.

V

SCHEUBEL'S LIFE AND WORKS

1. LIFE

Johannes Scheubel (Joannes Scheubelius, Scheybel) was born August 13, 1494, in Kirchheim, Württemberg.[1] Nothing is known about his family since the church register of Kirchheim goes back only to 1558, but he received his early education in his home town. Later he became a student at the University of Vienna, where the teaching of mathematics was greatly emphasized.[2]

Luther's Reformational ideas may have been the cause of Scheubel's leaving Vienna and going to Württemberg. But in order to continue his study of mathematics he changed to the University of Leipzig in the winter semester of 1532 and in March, 1535, he was a student in Tübingen.[3]

He acquired the degree of "Magister" in 1540, and on July 20, 1544, became "Docent of Mathematics." But at this time he was only an instructor of arithmetic and geometry.[4] In his published works of 1545 and 1549 he calls himself "Magister," but in his edition of Euclid (1550) the title page reads "Professor ordinarius." Evidently, he was in dire circumstances, since in 1551 and again in 1562 he asked the senate of the university for an increase in salary.[5] On February 20, 1570, Scheubel died in Tübingen and was buried there.[6] He left his instruments and

[1] J. C. Poggendorff, *Handwörterbuch zur Geschichte der exacten Wissenschaften*, 2 vols.; Leipzig, 1863–1904, vol. II, p. 791, "Scheubelius (Scheybl), Johann—Prof. d. *Mathematik an d. Univ. zu Tübingen, geb.* 1494, Aug. 18, Kirchheim, Würt., gest. 1570, Feb. 20, . . . "

[2] H. G. Staigmüller, "Johannes Scheubel, *ein deutscher Algebraiker des* XVI Jahrhunderts," *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Vol. IX, pp. 431–469 (Leipzig, 1899), "The question concerning the source of Scheubel's knowledge in arithmetic and in algebra is answered when we consider the excellent high school at Vienna," pp. 433–4; hereafter referred to as Staigmüller. This school is made famous in mathematical studies by Peurbach and Regiomontanus.

[3] *Urkunden zur gesch. der Universität Tübingen aus den Jahren 1746 bis 1550*, Tübingen, 1877, p. 658, "Jo. Scheubel-Scheybel, Scheyble-von Kirchheim u. T. insc. 1535, mag. 1546 soll in Wien und Leipzig studiert haben;" hereafter referred to as *Urkunden Tübingen*. "Diese Urkunden, Oberbibliothekar Prof. Dr. Rudolf von Roth veroffentlicht," Staigmüller, p. 435.

[4] "In der am 20 Juli 1544 von Herzog Ulrich erlassenen Ordnung der Artistenfakultät in Tübingen: 'Mit Maister Johan Scheublin soll gehandlet werden, das er umb ain bestimpte Besoldung Euclidem zu lesen, auch Arithmetices und Geometrie ler den Jungen einzubilden.'" *Urkunden Tübingen*, pp. 235–6.

[5] *Facultas Philos. F. Professorum vocationes electiones* I, 1510–1599, no. 29 and 29 *a*.

[6] *Württembergische Vierteljahrshefte für Landesgeschichte*, Stuttgart, 1920, p. 242 footnote, "In Scheckg's Leben (ben Adami) wird Scheubel's Tod in das Jahr 1570 gesetzt;" hereafter referred to as *Württembergische Landesgeschichte*.

mathematical manuscripts to the University.[1]

2. LIST OF PUBLICATIONS

a. De Numeris et Diversis Rationibus seu regulis computationum opusculum a Joannes Scheubelo compositum . . . Lipsiae . . .MDXLV, dedicated to the doctors and magistrates of Tübingen.

This arithmetic contains five chapters: "I. Counting with integral numbers as far as cube root; II. Counting with proportions and equations; III. Counting with common fractions; IV. Counting with 'physical fractions'; V. Proportion, extraction of roots, problems in arithmetical, geometrical and harmonic progressions, and arithmetical epigrams from the Greek anthology."[2]

b. Compendium Arithmeticae Artis—per Johannem Scheubelīū adornatum et conscriptum—Basel, 1549; dedicated to the young academic students in Tübingen.

This *Compendium* of 86 pages, a great improvement over the original 254 pages, was much appreciated and was permitted a second edition in 1560.[3]

c. Euclidis Magarensis, Philosophi et Mathematici excellentissimi, sex libri priores, de Geometricis principis, Graeci et Latini, una cum demonstrationibus propositionum. Algebrae porro regulae, propter numerorum . . . his libris praemissae sunt, . . . , dedicated to Anton Fugger and to the sons of Raimund Fugger, merchants of Augsburg.[4] "This work is Scheubel's masterpiece."[5]

[1] K. Klupfel, *Geschichte und Beschreibung der Universität Tübingen*, Tübingen, 1849, p. 496, "Die nach 1534 neu gegründete Universitätsbibliothek war hintschtlich ihrer Bermehrung eigenlich nur auf Geschenke angewiessen. So schenkte 1562 ein Professor Scheubel eine Anzahl aus Holz geschnittener mathematisher Figuren; ein gewisser Eberh. Schultheiss vermachte bald darauf seine Bibliothek." See also Andreas Christoph Zellern, *Ausführliche merckwürdigkeiten hochfürstlich der Württembergischen Univerisität und Stadt Tübingen*, Tübingen, 1745, p. 705; Staigmüller, p. 458; and Smith, *Rara*, p. 233.

[2] Smith, *Rara*, pp. 233, 236.

"This work is the production of a scholar rather than a man conversant with the demands of business. While Scheubel tried to write a mercantile arithmetic, the result was far removed from the needs of the common people. It carries the work in subjects like the roots so far that the ordinary Rechenmeister could not have used it. Moreover, it is written in Latin and is much more extended than the work of Gemma Frisius, so that it appealed neither to the business school nor to the ordinary classical school. A great deal of attention is given to exchange, the rule of three and the extracting of roots of high order. Attention is also given to problems which would now form part of algebra, and there is little treatment of geometry from the standpoint of mensuration.

"While Scheubel is not much appreciated to-day, he was really ahead of his time. He tried to banish the expression 'rule of three,' and to substitute 'rule of proportion.' His explanation of square root is in some respects the best of the century, and he dismisses with mere mention the 'duplatio' and 'mediatio' of his contemporaries. He extracts various roots as far as the 24th, finding the binomial coefficients by means of the Pascal triangle a century before Pascal made the device famous."

[3] G. J. Vossius, *De Universae Matheseos Natura et Constitutione Liber*, Amsterdam, 1650, p. 318.

[4] In the sixteenth century the Fugger family established one of the most powerful financial houses which the world has ever known. The fact that they were friends and benefactors of Scheubel is proof of his influence at this period. H. Pantaleon mentions Scheubel, together with the Fuggers, in his *Teutscher Nation Heldenbuch*, Basel, 1567, pp. 443–4, "Also hat er auch die Regel Algebrae so bistzhar den gelehrten undekandt/harfur gebracht/und mancherleiy instrument gesteller/zu welche in die herzen Fugger durch ir freygebigkeirbeholffer gewesen. Desizhalben is Scheubelius den gelehrten wohl bekandr/und sehr lieb gewesen . . .," p. 444; hereafter referred to as Pantaleon. The importance of the Fugger family at this time is shown by the *Fugger News-Letters, 1568–1605*, edited by Victor Von Klarwill, London, 1924.

[5] Staigmüller, p. 441.

d. Algebrae compendiosa facilisque descriptio, qua depromuntur magna Arithmetices miracula, Paris, 1551.

This algebra was first published as a preface of 76 pages to Scheubel's above publication of the six books of Euclid. Because of its great popularity it was permitted this separate edition.[1]

e. Jacobi Fabri Stapulensis in Arithmetica Boëthii epitome, una cum difficiliorum locorum explicationibus et figuris (quibus antes carebat) nunc per Joannem Scheubelium adornatis et adiectis, Basel, 1563, dedicated to the deacons and to the members of the Faculty of Arts of Tübingen.[2]

f. Das sibend, acht und neünt Buch des hochberümbten Mathematici Euclidis Megarensis, durch Magistrum Johann Scheubel, der löblichen Universitet Tübingen, Augsburg, 1558.[3]

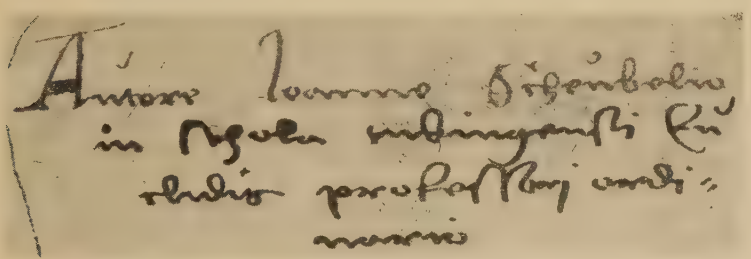
3. MSS. IN TÜBINGEN

In order to identify the Columbia University Manuscript, Professor Smith had photographs taken of the MSS. which he knew were in Tübingen University. When I wrote a letter of inquiry to this University concerning other reproductions of Scheubel's handwriting,[4] the photostatic copies sent to me verified Professor Smith's three photographs as well as the handwriting of our text.

a. Liber Primus

Angulos aequales in se habuerit: Rectus, est uterque aequalium angulos.[5]

[1] Staigmüller, p. 445, "Two of these reprints are in the British Museum."
[2] J. C. Heilbronner, *Historia Mathematica Universae a mundo conditio ad seculum post Chr. Nat.* XVI, Leipzig, 1742, p. 793.
[3] Staigmüller, p. 447, "Up to this time Scheubel had written for the young students, but now he selects for his audience 'the public who love the art of reckoning' and, therefore, he uses the German language."
[4] This sample of Scheubel's handwriting from the University of Tübingen is a direct verification that our manuscript is his work.



[5] This is evidently the last part of the 10th demonstration given in the first chapter of one of Scheubel's editions of Euclid. Each problem, given in Latin, is premised by a demonstration in Greek, demonstrated with excellent diagrams.

b. Euclidis Megarensis, Philosophi et Mathematici excellentissimi, Graeci et Latini Autore Joanne Scheubelio. . . . Praemissae autem sunt eis regulae Algebrae et ipsae demonstratae.[1]

c. Liber Primus.

Ex figuris autem Trilateris est . . . 23. Aequalaterum quid triangulum est. . . .

[1] "Nachdem H. Staigmüller's Abhandlung gedruckt war, hat H. Curtze in der Tübinger Bibliothek die Originalhandschrift von Scheubel's lateinischer Uebersetzung der sechs ersten Bücher Euklid's aber ohne die im Druck von 1550 vorausgehende Algebra aufgefunden," Cantor, *Geschichte*, II (2), p. VIII.

VI

SCHEUBEL'S INFLUENCE

1. IRRATIONALS AND APPROXIMATIONS OF ROOTS

Recognition of Scheubel's influence is to be found in Staigmüller's explicit statements, in which he asserts that his purpose is "To take out of almost oblivion a former teacher of Tübingen, and to give him the place as a mathematician, especially as an algebraist, which he justly deserves."[1] He cites many proofs of appreciation, given by Scheubel's contemporaries, of an author who greatly contributed to the development and teaching of arithmetic, algebra, and geometry.[2]

Scheubel's excellent treatment of irrationals is far superior to that given by any of his contemporaries.[3] He popularized the root symbols, ℒ, ȝℒ, and ȴℒ, which represent, respectively, square root, cube root and fourth root. Robert Recorde, the founder of the British school of mathematics, uses these symbols,[4] and gives practically the same definitions and development of the four fundamental operations with surd numbers[5] which Scheubel uses in his *Algebra* (1551) and also in his MS.[6] Indeed, the similarity is so marked that one is inclined to conclude that Recorde must have had Scheubel's *Algebra* before him when he wrote his book.

In both of his algebras Scheubel arranges the work with some attention to the teaching of the subject. He is a modern in that he begins with easy processes and gradually works up to more difficult applications. When he wishes to emphasize different rules he sums up his processes in a *compendium*. Most mathematicians of this time give rules with only a slight explanation

[1] Staigmüller, p. 432.
[2] Staigmüller, p. 438.
[3] J. Bohnenberger (Prof. of Math., Physics and Astron. in Tübingen, 1831), *Biography of Scheubel*, Cod. Hist. fol. 657. Univ. of Stuttgart; hereafter referred to as Bohnenberger.
[4] Robert Recorde, *Whetstone of witte*, London, 1557, fol. Ll 3, *v*. Scheubel says that this symbol is often accompanied by the symbol ra. and Recorde remarks, "Howbeit many times it hath with it, for the moare certaintie the *Cossike* sign, z.—$\sqrt{z}$. 20, the square root of 20."
[5] Recorde, *Whetstone of witte* (1557), fol. S1, *r*.
[6] Scheubel, *Algebra* (1551), fol. 25, *r*–31, *v*; MS. pp. 15–39 (our text). This work includes surds of the second, third and fourth quantities.
Recorde, *Whetstone of witte* (1557), fol. Ll 3, *v*; Qq 3, *v*.

and without following them with proper reasoning and application as Scheubel does.

Both Scheubel's and Recorde's definitions of binomials and residuals are taken from Euclid's *Elements*, X, 36 and 37.[1] In their development they include problems involving the four fundamental operations. Scheubel gives some geometrical applications.[2]

His many applications of the following formulas include both whole numbers and fractions:

$$\sqrt{a}.\sqrt{b} = \sqrt{ab}^{3} \quad \sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

$\sqrt{ka^2} + \sqrt{kb^2} = \sqrt{k(a+b)}$,[2] if all roots of the sum are commensurable.

$$\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm \sqrt{4ab}}; \quad \frac{m \pm n}{p \pm q} = \frac{(m \pm n)\ (p \mp q)}{p \pm q \quad (p \mp q)}$$

Scheubel gives this general rule for extracting the root of any binomial or residual, $a \pm b$, where one or both parts are surds and where a is the greater quantity:

$$\sqrt{\frac{a + \sqrt{(a^2 - b^2)}}{2}} \pm \sqrt{\frac{a - \sqrt{(a^2 - b^2)}.}{2}}^{4}$$

The following general formula is used for the extraction of roots:

If

$$\sqrt{a} > \sqrt{b}, \quad \sqrt{\sqrt{a} \pm \sqrt{b}} = \sqrt{\tfrac{1}{2}\sqrt{a} + \sqrt{\frac{a-b}{4}}} \pm \sqrt{\tfrac{1}{2}\sqrt{a} - \sqrt{\frac{a-b}{4}}}^{5}$$

His explanation of square root is in some respects the best of the century.[6] When the root is irrational, he uses the following methods of approximation:

[1] Scheubel MS. (our text), pp. 32–37; Recorde, *Whetstone of witte* (1557), fol. Pp. 4, r, seq.
[2] Note 1, p. 33, of our text.
[3] "The following calculation forms approach the modern symbolism for the product of binomials. $2\sqrt{9} = \sqrt{4.9} = 6$. This is the first approach to an algebraic treatment in systematic form of surd quantities," Karpinski, *al-Khowârizmî*, p. 95.
[4] "This rule will only succeed, however, so as to come out in simple terms, in certain cases, namely, either when $a^2 - b^2$ is a square, or when a and $\sqrt{(a^2 - b^2)}$ will reduce to a common surd, and unite: in all other cases the root is in two compound surds, instead of one," Chas. Hutton, *Tracts on Mathematical and Philosophical Subjects*, 3 vols., London, 1812, Vol. II, p. 243; *Mathematical and Philosophical Dictionary*, London, 1796, p. 179; hereafter referred to as Hutton, *Tracts*.
[5] See Scheubel's development, pp. 37–39 of our text. Smith, *Rara*, p. 236. "In his *Arithmetic* (1545), Book V, he extracts various roots as far as the 24th, finding binomial coefficients by means of the Pascal triangle, a century before Pascal." Recorde, *Whetstone of witte* (1557) fol. R *r* 3, *r* seq.
[6] Smith, *Rara*, p. 235.

If $a^2 \angle a^2 + b \angle (a+1)^2$, $\sqrt[2]{a^2+b} \sim a + \dfrac{b}{2a+1}$ [1]

If $a^3 \angle a^3 + b \angle (a+1)^3$, $\sqrt[3]{a^3+b} \sim a + \dfrac{b}{3a^2+3a+1}$

2. PROBLEMS AND TREATMENT OF EQUATIONS

In his algebras Scheubel introduces his characters, which, beginning with the zero power, can be extended to any power,[2] and, after stating his rules for the plus and minus signs, he gives rules and applications for the four fundamental operations. He was one of the earliest to consider these operations as we do now, with whole numbers and fractions. They had been neglected before because algebra was considered as a study for mathematicians, and not for boys and girls in their school years.

Without doubt, Scheubel was influenced by Stifel in many ways, but a comparison of their algebras shows a great deal of originality on Scheubel's own part.[3]

He develops and illustrates the following three types of equations: The first contains different powers of the unknown as in:

$$ax^m = bx^n$$ [4]

The second contains three characters, one following after another as in:

$$ax^{m+1} + c = bx^{m+2}$$ [5]

The third contains characters which do not follow each other immediately but at equal intervals as in:

$$ax^{m+2n} + bx^{m+n} = cx^m$$ [6]

Scheubel is very skillful, also, in getting around the introduction of several unknowns. In his three methods of solution for the following problem, he first takes the whole number as the unknown, and then the first and second parts:

A number is to be divided into three parts, such that the first part is to the whole as 2:3, the second part is to the whole as 1:2 and the third part, plus 4, is to the whole as 3:4. Find the number and its parts.

[1] The first use of the equality sign is found in Recorde's *Whetstone of witte* (1557), fol Ff 1, *v.* Previous to that time ∼ was used.

[2] Scheubel, fol. 2, *r*, of his 1551 edition, gives the generally accepted characters, used also by Recorde, *Whetstone of witte*, fol. S3, *r*. See page 3 of our text for both of Scheubel's representations of characters.

[3] "*Algebrae compendiosa facilisque descriptio* (1551). Scheubel weicht zwar in der Bezeichnung von der Stifel's ab, zeigt sich aber sonst vielfach von diesem beeinflusst, doch so, dass immerhin noch von eigener Arbeit die Rede sein kann." Treutlein, *Die Coss*, p. 20.

[4] See pages 39–49 of our text.

[5] See pages 49–57 of our text.

[6] See pages 57–67 of our text.

His results are interesting:

"Impossibile, cum tertia, pars nihil sit, propterea quod duabus prioribus totum et plus etiam conveniat. Vel facit: Totus numerus $4\frac{4}{11}$. Partes vero: prima $2\frac{10}{11}$; secunda $2\frac{2}{11}$; tertia $-\frac{8}{11}$." [1]

In the last part of his *Algebra* (1551) he gives the solution of a cubic equation. Since his method is not given he may have solved it through experimentation.[2]

Scheubel marks the beginning of that period of mathematical history in which algebra approached the modern form. Contemporaneously with the remarkable discoveries of the Italian mathematicians, algebra was increasing in popularity in France, in England and in Germany, where the works of Scheubel and Stifel are noteworthy for their perspicuity and for the introduction of a more complete symbolism for quantities and for operations.[3]

Even in his own day, Scheubel's influence had penetrated to foreign countries, especially to France. Peletier speaks of Nunez and Riese but questions whether their books had ever been published. Scheubel, whose work he greatly admired, he used to a considerable extent.[4] But while his representations of characters, his treatment of irrationals, his extraction of roots, his equations and his problems resemble Scheubel's processes, they do not have Scheubel's scholarly development or appropriate applications. And Peter Ramus says that when he visited Europe and called upon all the great mathematicians he preferred the Germans as a whole to all others and that Scheubel should be honored above all his contemporaries as the pride of Tübingen.[5] A little later, Rafael Bombelli, the most important Italian mathematician of the latter part of the sixteenth century, mentions first Scheubel, and then Stifel, as algebraists worthy of the name.[6] In England also, Recorde's *Whetstone of witte* (1557), the first English edition of an algebraic text, is largely a repetition of Scheubel's book, although he combines the latter's three types of equations into two.[7]

John Wallis, a seventeenth century mathematician, includes

[1] Staigmüller, p. 456.
[2] Bohnenberger's *Biography of Scheubel.* Staigmüller, p. 459.
[3] *Encyclopædia Britannica,* Eleventh edition, vol. I, p. 619 (Cambridge, 1910).
[4] Jacobi Peletarii, *Libri Duo,* Parsis, 1560, pref. 4 (4, *r*), and Chap. I, p. 2 (b I, *v*), L'Algebra De Jaques Peletier, 1554.
[5] *Württembergische Landesgeschichte,* p. 243; note, p. 242. "Scheubel was excellent in analyzing." Ramus also calls him a monument of genius. P. Rami, *Scholarum Mathematicarum libri unus et triginta,* Lib. XXXI, Basel, 1569, p. 66 (lib., II).
[6] Rafael Bombelli, *L'Algebra Opera,* Bologna, 1579, p. d (3) *r.*
[7] Recorde *Whetstone of witte* (1557), fol. Ff. 3, *r,* "Now will I shewe you the varities of equations, taught by Scheubelius, bicause you maie perceive how thei bee conteined in those 2 forms, named by me."

Scheubel in his list of important algebraists of the preceding century.[1] And Hutton characterizes the *Algebra* (1551) as "a very clear though succinct treatise which both in form and matter much resembles a modern printed book."[2] Bohnenberger likewise calls attention to the fact that Scheubel's very important algebraic demonstrations of geometry were unusual and very elaborate for his time, and that his treatment of equations is exceptional. (In his second equation, Scheubel uses the modern method of completing the square; in his illustrations of the third equation he clears the equations of fractions.) He further comments on Scheubel's ingenious dexterity in his types of problems. (For the greater part, Scheubel's different methods of solutions are arranged in logical steps.)

Scheubel's success as a teacher was also noteworthy.[3] Indeed, all of his works, except the last, were the result of his academic instruction and were written for his academic instruction. He wished to make algebra easy and interesting and, therefore, attempted to use vivid processes which would make the subject applicable to the ordinary business affairs of his day. Thus he hoped to unite practical life and academic instruction[4] in presenting such types of problems as: The partnership problem, the general and his army, a comparison of men's money, the question of silk, the strange journey, and problems about gold coins and the exchange of money. The fact that Recorde used practically these same types of problems is another proof of Scheubel's predominating influence in the *Whetstone of witte* (1557).[5] Moreover, much of the terminology of elementary mathematics goes back to Recorde's[6] work and a great part of this terminology he also got from Scheubel.

No doubt Scheubel was somewhat influenced by Stifel, but when the algebras of these two are compared, one is conscious of a marked contrast.[7] The subject matter is to a great extent similar, but Stifel's presentation is somewhat antiquated, while Scheubel's is modern.

His presentation is modern in its logical development of sub-

[1] Wallis, *Algebra*, p. 63.
[2] Hutton, *Tracts*, II, pp. 241–2.
[3] Pantaleon, *Teutscher Nation Heldenbuch*, pp. 443–4.
[4] Staigmüller, p. 450.
[5] See pages 45 and 54 of our text for a comparison of problems. See *Whetstone of witte*, fol. Hh 2, *v.*, and fol. Ll 3 *v.*
[6] *Bibliotheca Mathematic*, Leipszig, 1912–13, p. 228.
[7] Smith, *History*, I, p. 329.

ject matter, the arrangement of material for teaching purposes, and the decisiveness of the conclusions.[1] His rules are clear-cut and are followed by appropriate examples, and, because these examples are practical, he easily arouses the interests of his students. Scheubel was a modern also in that he was "concerned with the teaching of mathematics for the soul of the subject as well as its use in trade and in the shop, he had to meet the opposition of those who would teach only that which is immediately useful."[2]

[1] Recorde valued this last trait to such an extent that he frequently copies it.
[2] Smith, *Mathematics*, pp. 156–57.

VII

THE SCHEUBEL MS.

1. GENERAL APPEARANCE

Codex Universitatis Columbiae: Columbia University Library MS. X 512, Sch. 2, Q. 308 pages; Latin MS. in German hand, c. 1550.

Pages 1–68, *Breuis ac dilucida regularum Algebrae descriptio, autore Joanne Scheubelio, in inclyta Tubingensi academia Euclidis professore ordinario*—our text. This contains two problems which are found in Scheubel's algebra of 1550, a published preface to the first six books of Euclid, *Euclidis Megarensis, Philosophi Mathematici excellentissimi, sex libri priores . . . Algebrae porro Regulae, propter numerorum exempla, passim propositionibus adiecta, his libris praemissae sunt, eademque demonstratae* (Basel, 1550). A separate edition appeared in Paris in 1551. *Algebra compendiosa facilisque descriptio . . .*

Pages 69–70 blank.

Pages 71–122, *Liber Algebrae et Almucabola continens demonstrationes aequationum regularum Algebrae,* edited by L. C. Karpinsi, *Robert of Chester's Latin Translation of the Algebra of al-Khowarizmi,* New York, 1915.

Pages 123–157, *Addita quaedam pro declaratione Algebrae* by Scheubel. The explanation of the above algebra was published in Professor Karpinski's book.

Page 158 blank.

Pages 159–308, *Liber Jordani Nemorarii de (numeris) datis in quatuor partes digestus.* This is not the complete text of the *De Numeris Datis,* but it contains a statement of the problems and their solutions according to the rules of algebra. In these solutions Scheubel introduces the use of *N* for number, co. for cosa, the first power of the unknown, and the signs, + and −.

2. HOW ACQUIRED BY COLUMBIA UNIVERSITY

Professor Smith accidentally found it in a European book catalogue and ordered it for the Columbia University Library.

3. PROOF THAT IT IS SCHEUBEL'S WORK

The peculiarity of the handwriting and the name *Scheubel* on the first page led Professor Smith to believe that the acquisition might be an unpublished MS. of this mathematician. When he had photographs taken of the various Scheubel MSS. which he knew were in the University of Tübingen, and compared the handwritings, he realized that his conjecture was true.

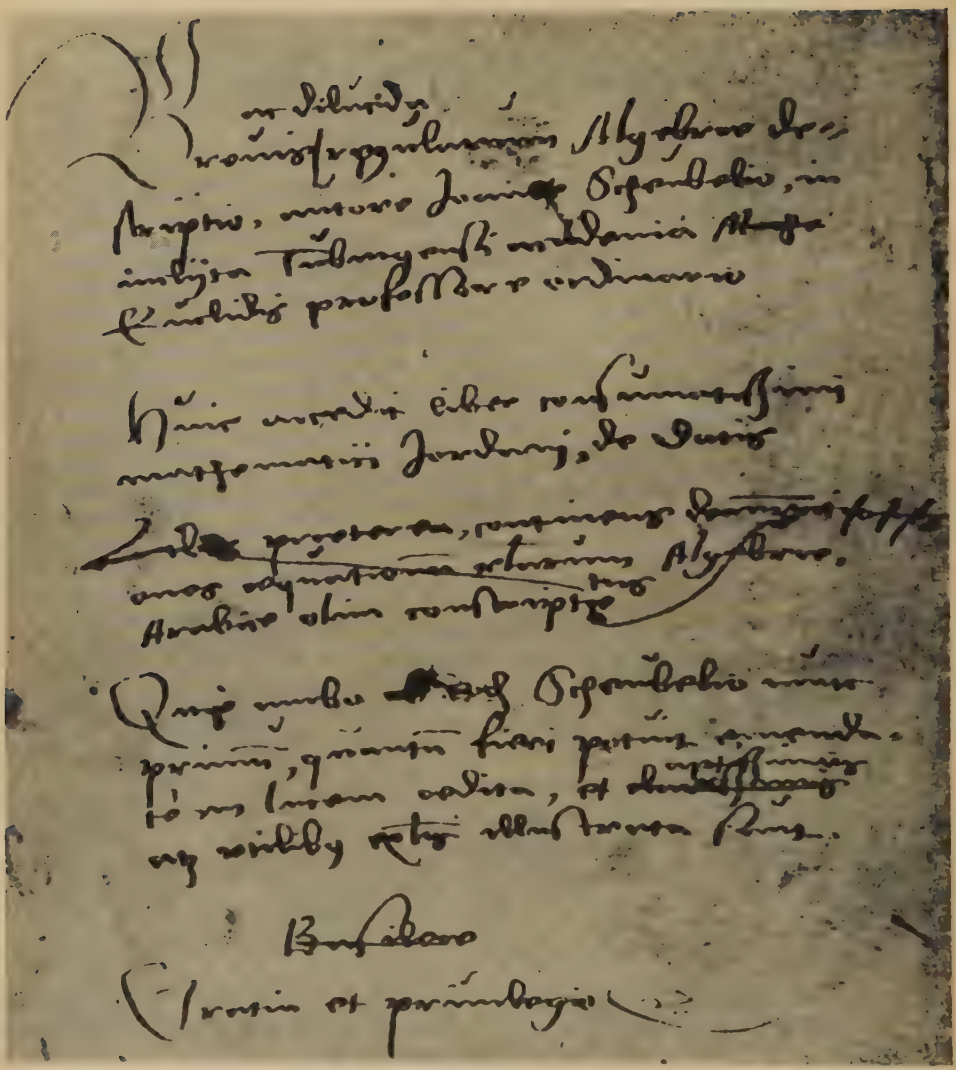
When I compared this MS. with Scheubel's *Algebra* (1551), *Algebra compendiosa facilisque descriptio . . .* , I found that he had included here two of the same problems which he had used in his printed edition. His rules for the fundamental operations are practically the same in both works, but in the MS. he gives a logical development of the operations involved in whole numbers, fractions, surds and binomials, and follows these with a second treatise involving three types of equations, together with practical illustrations of these three types, while in his algebra he changes the order of development, gives a longer treatment of surds, which occupies one-third of the book, and closes the work with four pages of Greek epigrams. The same scholarly treatment is used in both of the algebras.

4. POSSIBLE HISTORY

The presence of marginal notes suggests that the MS. was prepared for publication; *Textus*, for example, is written to indicate the statement of the problem, *Minor*, an explanation. Since Scheubel had bequeathed all of his MSS. to Tübingen, presumably his work was in the printer's hands at the time of the author's death or a little later, but for some unknown reason, possibly because of the wretched chirography, was never printed. It thus became separated from the other MSS. and was not identified for nearly four centuries when Professor Smith purchased it for Columbia University.

VIII

PHOTOGRAPHIC COPIES OF ORIGINAL SCHEUBEL MANUSCRIPT WITH TRANSLATION AND NOTES



PAGE 1

A Brief and Clear Description of the Rules of Algebra[1]

BY THE AUTHOR, JOHANN SCHEUBEL

ORDINARY PROFESSOR OF EUCLID IN THE ILLUSTRIOUS ACADEMY OF TÜBINGEN[2]

To this is added the *De* (*Numeris*) *Datis*[3] of Jordanus, an excellent mathematician. A demonstration of the rules of algebraic equations, originally written in Arabic,[4] is also added. Both of these are now for the first time revised and illustrated with appropriate and useful examples by Johann Scheubel.[5]

AT BASEL

With Favor and Privilege

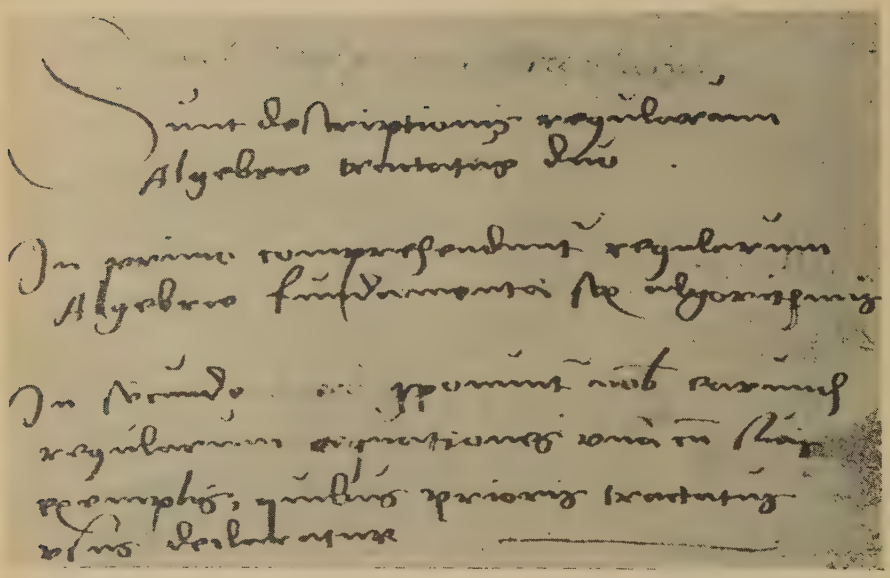
[1] The above title of the manuscript can be compared with his printed work of 1551, "Algebrae compendiosa facilisque descriptio, qua depromuntur magna Arithmetices miracula, Pariis, 1551. Authore Joanne Scheubelio, Mathematicarum professore in academia Tubingensi," which was first published in 1550 as a preface of 76 pages to his six books of Euclid, "Brevis Regularum Algebrae Descriptio, una cum demonstrationibus Geometricis. Euclidis Megarensis, Philosophi et Mathematici excellentissimi, sex libri priores, de Geometricis principiis, Graeci et Latini una cum demonstrationibus propositionum. Algebrae porro regulae propter numerorum exempla, passim propositionibus adiecta, his libris praemissae sunt, eademque demonstratae . . . Autore Joannes Scheubelio in inclyta academia Tubingensi Euclidis professore ordinario, Basel, 1550. Cum gratia et privilegio."

[2] "Annus quo coepit exorsum Academiae Tubingensis 1477," *Catalogus Academiae Tubingensi*, Tübingen, 1541. This university proved to be the source of inspiration for Scheubel's literary activities while he was professor of mathematics.

[3] Pp. 159–308 of this MS.

[4] Pp. 71–122 of this MS, "Liber Algebrae et Almucabola continens demonstrationes aequationum regularum Algebrae" by al-Khowârizmî (c. 825), translated by Robert of Chester. (c. 1140 A. D.), the Latin translation was published by Louis Charles Karpinski, New York, 1915.

[5] Scheubel's "Additions to Algebra," pp. 123–157, show clearly that because of his knowledge of other Latin translations of al-Khowârizmî, such as Libri's, he could include in the contemplated publication of this MS. more than Robert of Chester had given.



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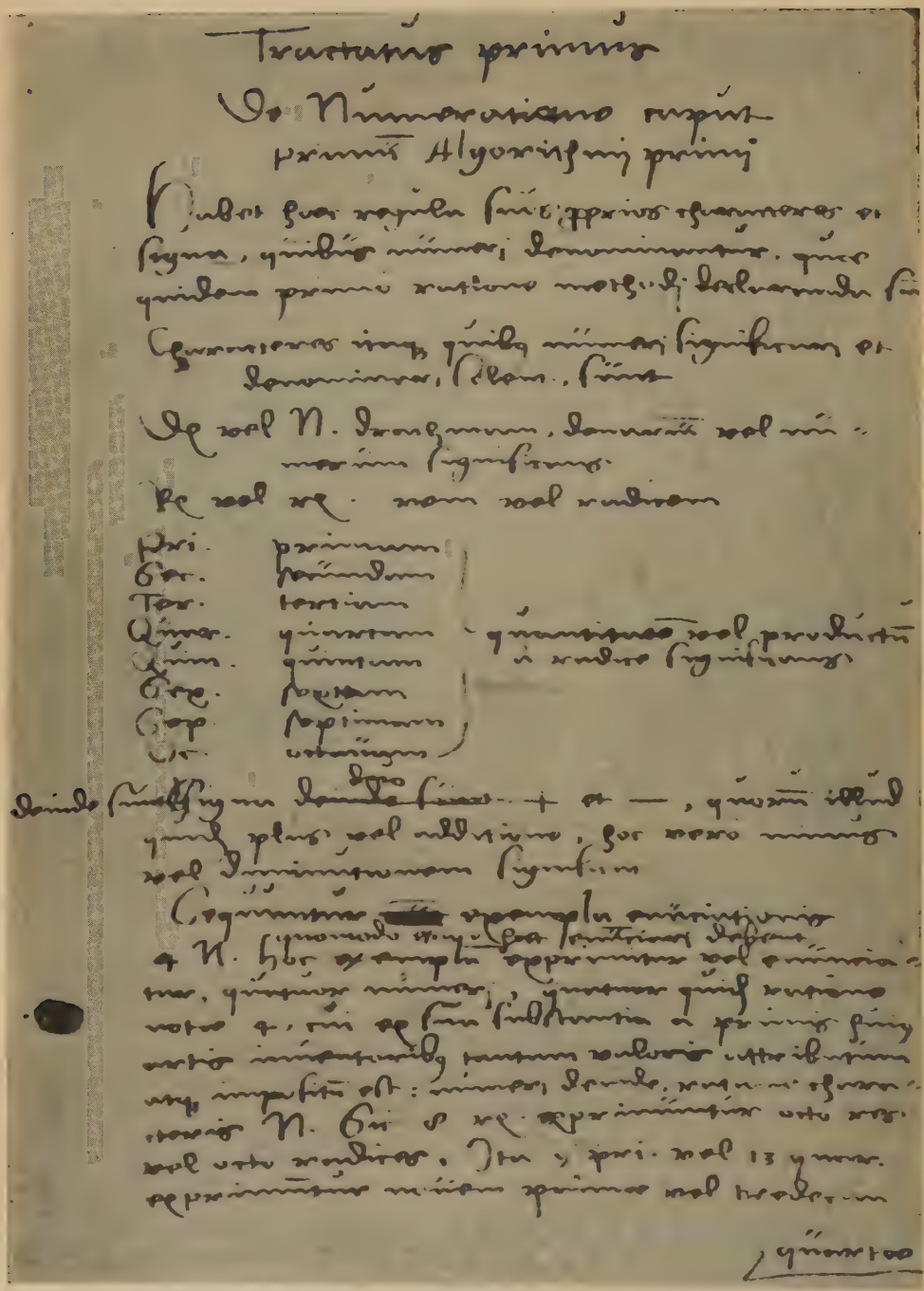
This book contains two treatises of exposition of the rules of algebra:[1] The first presents the fundamentals of the rules of algebra or algorism. The second sets forth equations under these rules, together with certain examples, wherein their use is demonstrated.

[1] Scheubel prefaces his *Algebra* (1551), fol. 2, r, with the following: "Porro harum regularum inventionem ascribunt Diophanto Graeco scriptori, qui, ut author est Regiomontanus in praefatione Alphragani, libris tredecim eas descripsit, atque ut Latini Rei et Census, sic Arabes regulas illas vocabulo suo appellare solent Algebras, id quod obiter indicandum erat."

In his "Additions of Algebra," pp. 123–157 of this MS., he states the six canonical rules for the solution of equations, but on page 39 of our text he states that equations are divided into three classes.

Pacioli (1494, ed., fol. 67, r) calls algebra the greater art as compared with arithmetic, the lesser art: "Per loperare de larte magiore: ditta dal vulgo la regola de la cosa ouer algebrae amucabala." Treutlein, *Die Coss*, p. 3, expresses the same idea in the following: "Welchen man am Ausgange des Mittelalter im Gegensatze zur niederen oder gemeinen Arithmetik (Ars minor) als die höhere Arithmetik (*Ars major*, Arte maggiore) oder Algebra . . . als die Coss bezeichnete."



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NUMERATION[1] OF THE FIRST ALGORISM

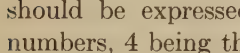
Chapter I

This rule has its own particular characters and signs by means of which the numbers are indicated and which are explained by the first rule of computation. The characters by means of which the numbers are usually denoted and indicated are *D* or *N*, signifying drachma,[2] denarius[3] or number[4] and *R* or *r*, the thing (unknown) or root (radix).[5]

Pri.	$= x^2$		Quin. $= x^6$
Sec.	$= x^3$	Representing quantities	Sex. $= x^7$
Ter.	$= x^4$	or products of roots	Sep. $= x^8$
Quar.	$= x^5$		Oc. $= x^9$

Next there are the two signs[6] + or −, the former signifying plus or addition, the latter signifying minus or subtraction.

The following examples show how each of these statements should be expressed.[7] 4*N*, by which is expressed or stated 4 numbers, 4 being the number for four, this being the value given to it by the inventors of this science, and *N* indicating some particular number. Then 8 *re.* represents 8 things or roots ($8x$) and 9 pri. or 13 quar. represents 9 of the first ($9x^2$) or 13 of

[1] On page 2, *r*, of the 1551 edition we read: "Cap. I, Numeration. Characteres vocabulorum seu appellationi, quibus in his regulis numeri naturali quodam ordine proportionis denominantur, sunt,  etc."

Then on page 2, *v*, *Ra* or Radix is given for this notation.

[2] Scheubel's *Algebra*, 2 *v*, "O, quidem Numerum." The Greek drachma was a small weight (originally a "handful"). This was used by the ancients as both a weight and in case of the talent as a unit of value. In modern Greece it is a coin with the same value and rate of exchange as the franc (Smith, *History*, II, p. 36).

[3] The denarius was a Roman coin of 16 asses (Smith, *History*, II, p. 208).

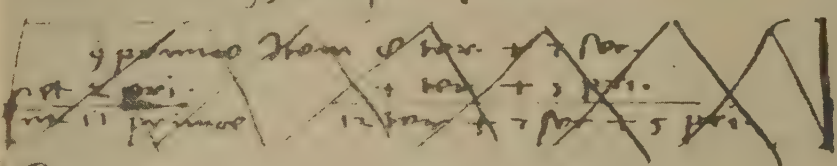
[4] Jordanus (c. 1225) in his *De Numeris Datis* says, "Numerus datus est cuius quantitas nota est." The Arabs used *mal* for the second power (meaning "wealth"), whence the medieval Latin *census* ("evaluation of wealth") was used, as well as the Italian *censo* and the incorrect *zenso* (Nesselmann, *Algebra*, pp. 55, 56). Diophantus (c. 275 A. D.) uses the word *dynamis* or power. See Scheubel's *Algebra*, p. 2, *v*. Karpinski's *al-Khowarizmi*, p. 82, line 5, has *res* for the unknown quantity.

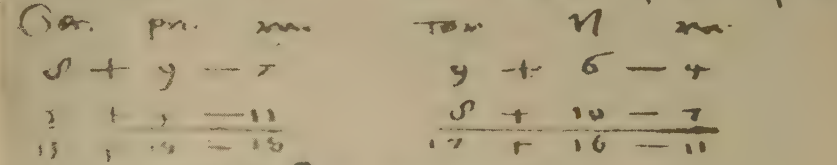
[5] Fibonacci, *Scritti*, II, p. 223. "Hoc est radicem rei in radicem 1/12 rei." Algebra was not uncommonly called *Ars Rei*, Nesselmann, *Algebra*, pp. 55, 56. The Arabs used *sha* (*sei*, *chai*, "thing" or "anything"). Their use of *jidr* (*dyizr*, root) results in the Latin *radix*.

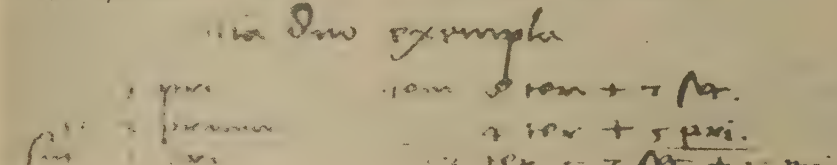
[6] Pacioli (1494) uses *m* for minus in his algebra, but he uses *de* for "6 de 10." Cardan (1539) writes R̸. V. 7. p. R̸. 4 for $\sqrt{7} + \sqrt{4}$. Tartaglia (1556) and Cataneo (1546) write *piu* and *men* for plus and minus in the Rule of False Position. Peletier (1549) uses the French plus and moins; the signs + and − are characteristic of Germany, but not before the 15th century. Johann Widman (1489) first used them to express excess and deficiencies in algebra. Vander Hoecke (1514) first used them as signs of operation.

[7] Scheubel's *Algebra*, fol. 2, *v*, "Quum ut in absolutis numeris naturali quodam ozdine maior sequitur minorem, ita quoque in denominatis proportione aliqua."

Regulæ Algebræ







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the fourth quantity ($13x^5$). Similarly, 20 sec. + 8re. represents 20 of the second quantity plus 8 things or roots ($20x^3 + 8x$); and 13pri.+ 9re.− 16*N* represents 13 of the first quantity plus 9 of the second quantity minus 16 ($13x^2 + 9x - 16$).

From the above examples anyone can readily perceive how easy this notation is, and that examples of this kind are commonly stated in the simple way in which we ordinarily express money, weights, or measures,[1] and also minutes in astronomy.[2]

Chapter II

Addition[3]

In the work in addition, the numbers are written with their characters and signs just as they are usually written in the common treatment of numbers. When a line is drawn under the columns, numbers of the same character are written by themselves, and their sum is written together with the appropriate character and sign in the proper column under the line. The addition is then completed.

Examples

$$\begin{array}{r} 8x^3 + 9x^2 - 7 \\ 5x^3 + 5x^2 - 11 \\ \hline 13x^3 + 14x^2 - 18 \end{array} \qquad \begin{array}{r} 9x^4 + 6 - 4x \\ 8x^4 + 10 - 7x \\ \hline 17x^4 + 16 - 11x \end{array}$$

$$\begin{array}{r} 9x^2 \\ +\ 2x^2 \\ \hline 11x^2 \end{array} \qquad \begin{array}{r} 8x^4 + 7x^3 \\ 4x^4 + 5x^2 \\ \hline 12x^4 + 7x^3 + 5x^2 \end{array}$$

[1] The business and commercial arithmetics, very popular during this period, used these terms in the sixteenth century.

[2] Stifel, *Arithmetica Integra* (1544), fol. 64, uses *astronomica progressio*, the natural "astronomical fractions," 1, 1/60, 1/3600, . . .

[3] Scheubel's *Algebra* (1551), p. 3, *v*, gives practically the same rule. Clavius, *Algebra*, Rome, 1608, fol. 16, introduces a chapter "De additione et subtractione numerorum cossicorum," which includes one of the earliest modern developments of the fundamental operations. Pacioli (1494), *Süma*, fol. 3, *v*, begins his work on algebra by considering a few definitions, then the laws of signs, and then the operations with monomials. Polynomials are considered only in such cases as they happen to arise in problems (fol. 127, *v*, seq.). Robert Recorde, *Whetstone of witte* (1557), fol. S 3, *r*, "The cossike nomber represents so many of that cossike denomination as ther be unities in that nomber. Add the nombers and put the same cossike denomination."

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In the last example, since neither thefirst quantity was found in the upper row nor the second quantity in the lower row, we wrote both the first and second quantities under the line, together with the numbers and signs ordinarily given to them. (Similarly),

$$\begin{array}{l} 9x^6 + 8x^3 - 7x \\ 8x^5 - 5x^4 + 4 \\ \hline 9x^6 + 8x^5 - 5x^4 + 8x^3 - 7x + 4 \end{array}$$

These examples are sufficient to illustrate numbers with the same name and sign.

But when numbers of the same character have different signs, if one number has the sign plus, +, and the other number has the sign minus, −, they are unequal (numbers are to be considered equal when they have the same characters and signs); then the excess of the greater over the less together with the name and sign of the greater is written below the line.

Example

$$\begin{array}{r} 7x^2 + 8x - 3x^3 \\ 5x^2 - 5x + 4x^3 \\ \hline 12x^2 + 3x + \quad x^3 \end{array}$$

Chapter III

SUBTRACTION[1]

In subtraction (the number) to be subtracted is written under the column from which it is to be subtracted. When a line is drawn underneath, the numbers of the separate denominations in the subtrahend are to be subtracted from the numbers of the same denomination in the minuend. Then when the remainders, together with their characters and signs,

[1] In Scheubel's *Algebra*, fol. 4, *r*, just before the discussion of subtraction, we find a "Tabula Comprobationis" in which he takes $2\frac{1}{3}$, $3\frac{1}{3}$, 7, etc., as roots and gives their powers up to and including the ninth power.

Regula Algebre

…nus, subtrahendum subtrahere possumus: et subtrahendo possumus addere. Et in omnibus multiplicationibus signorum + et −, nota per alia quod subtrahendi sit et quod non, quemadmodum et in additione subtrahendi diversitas facit, quando diversi sunt. Et quemadmodum subtrahendo nihil ei addimus, sic subtrahendo nihil ei subtrahimus: quare nec multa nec pauca.

Exemplorum

Primum

9 Car.	− 4 Pos.	+ 11 N.
4 Car.	+ 5 Pos.	+ 4 N
5 Car.	+ 3 Pos.	+ 2 N

Gradus				Terminus		
Car	Pos.	Nu.	N	Car	Pos.	N
1	+ 9	− 11	+ 8	13	− 5	− 4
4	+ 15	− 4	− 5	8	+ 8	− 11
3	− 6	− 2	+ 13	5	− 13	+ 2

Quantitas quae in signo minuendi
non reperta fuit, subtrahenda

Quidquid in toto addito, vel in eo quod subtractum fuit, vel in eo quod subtrahitur, numerus fuerit, cuius signum non reperitur; tunc in subtrahendi parte numerus ille cum suo signo in numero apposito: in alterno vero addito omnia ita ut numerus ignotus et signum. Sub sua subtrahitur

Secundum

A 4 Quadrata + 5 positis
subtrahitur 5 Quadratis + 4 N.
manet 4 Car − 5 Pos. − 4 N

Aliud

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are put in their place under the line, the subtraction is completed. Now special consideration must be given to the signs + and −, by means of which we recognize (the subtrahend), how much there is to be subtracted, as well as (the minuend), how much there is in the beginning, and also the remainder. If (the rules for these signs) are understood and applied, subtraction will be easy, but otherwise it will be difficult.

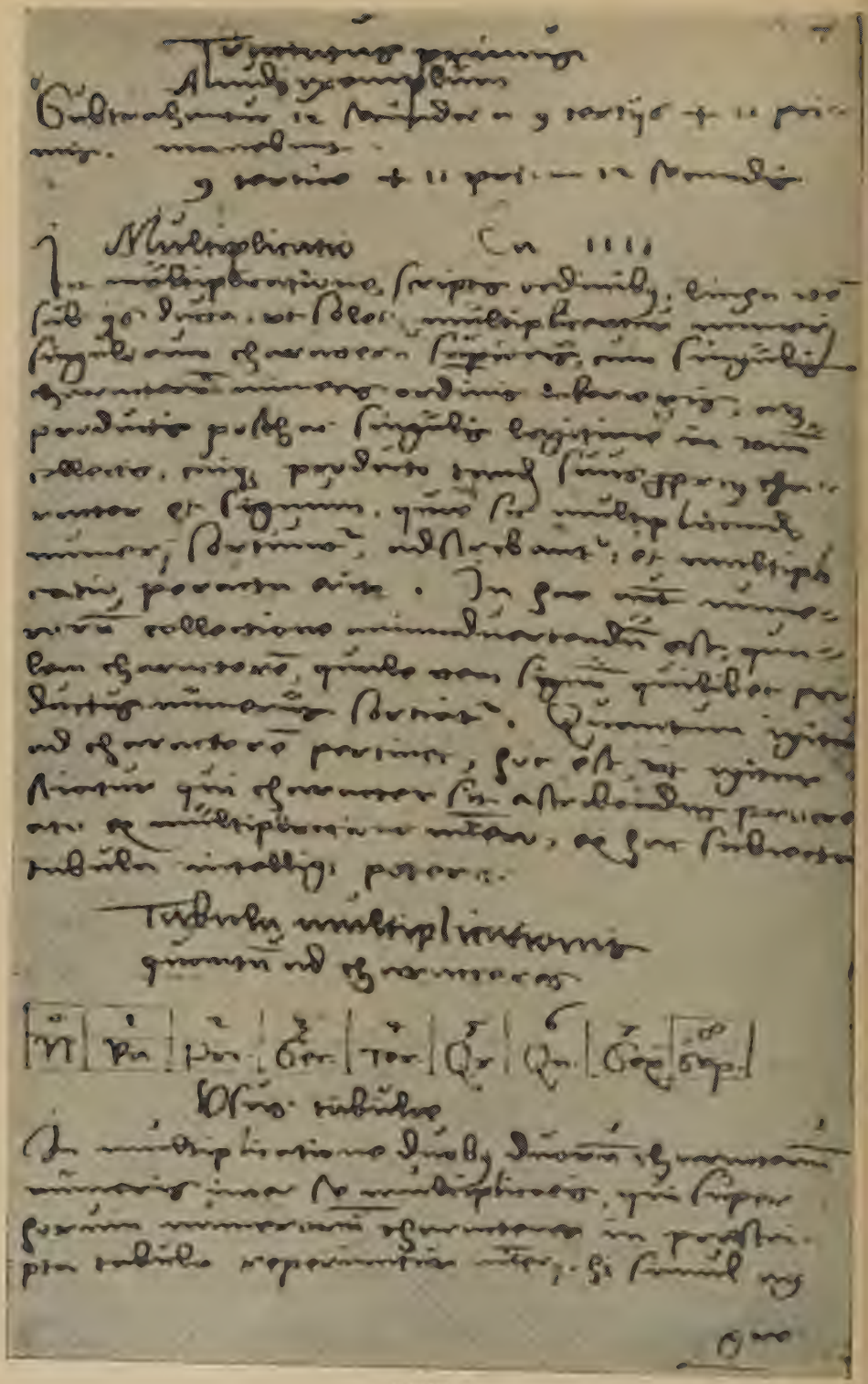
Examples

First	Second	Third
$9x^3 + 8x^2 + 11$	$7x^3 + 9x^2 - 11x + 8$	$13x^4 - 5x^2 - 9$
$4x^3 + 5x^2 + 9$	$4x^3 + 15x^2 - 9x - 5$	$8x^4 + 8x^2 - 11$
$5x^3 + 3x^2 + 2$	$3x^3 - 6x^2 - 2x + 13$	$5x^4 - 13x^2 + 2$

So far our illustrations have emphasized the treatment of signs. But (often) in one row, in either the subtrahend or the minuend, we find a number whose character is not repeated; then this number, if it is in the subtrahend, is written below the line together with its character and an opposite sign; but everything in the minuend, including the number, character and sign, should be written under the line.

Example

From	$9x^3 - 5x^2$
Subtract	$5x^3 + 9$
Remainder	$4x^3 - 5x^2 - 9$



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Another Example

Subtract $12x^3$ from $9x^4 + 11x^2$; the remainder is $9x^4 + 11x^2 - 12x^3$.

Chapter IV

Multiplication

In multiplication, when the rows are written and a line is drawn under them, the numbers of the single characters of the upper row are multiplied as usual by the numbers of each character of the lower row. When the single products[1] are collected and the appropriate characters and signs which result through multiplication of numbers are written for each product, then the multiplication is completed. The character and sign belonging to each number produced must be determined.[2] The following table will indicate the value which should be assigned to a number resulting from multiplication.

Table of Multiplication in Regard to Characters

0	1	2	3	4	5	6	7	8
1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8

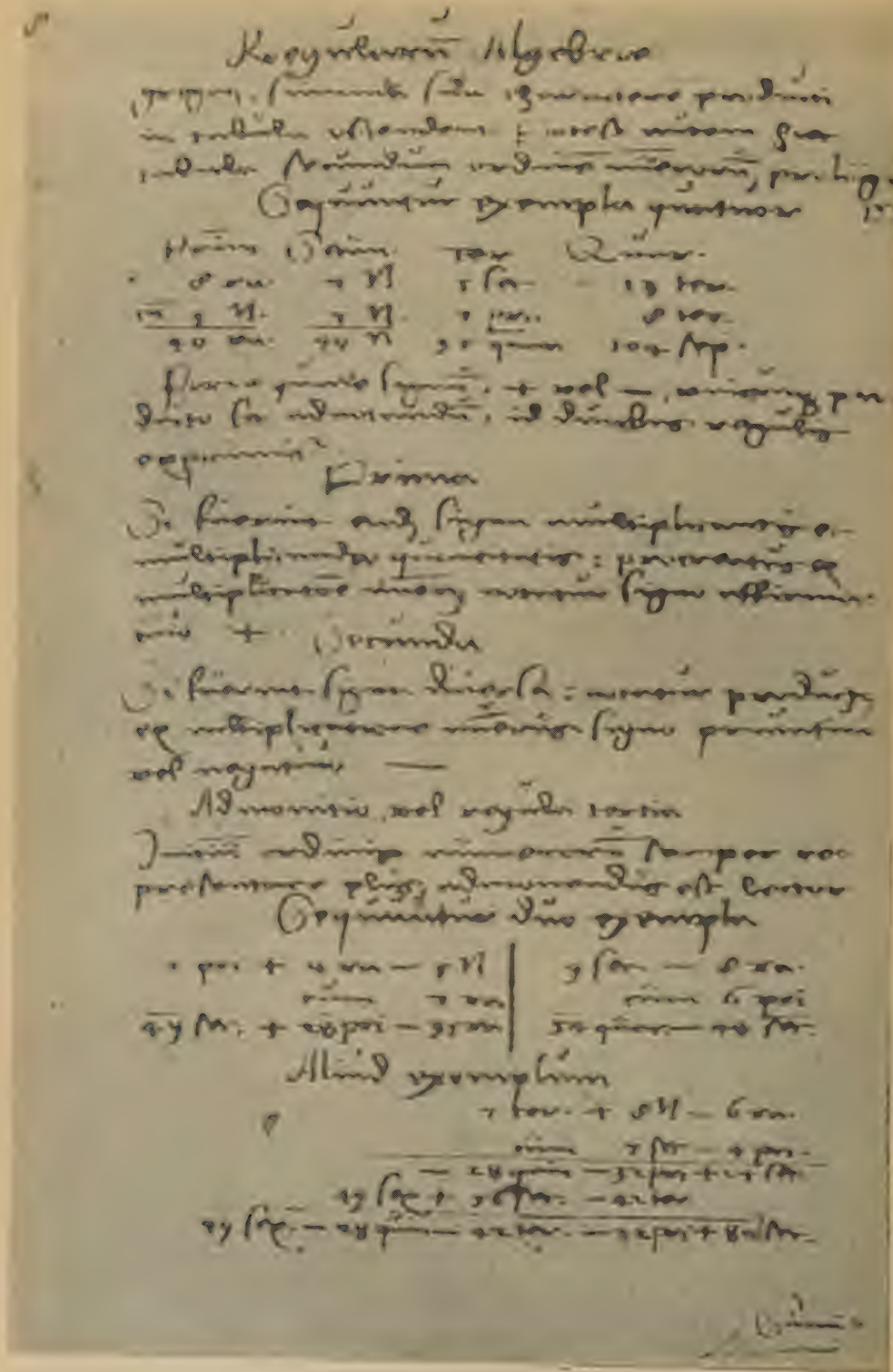
Another Table

Multiply (the coefficients), the two numbers which belong to the two different characters,[3] and if the sum of (the exponents), the numbers above the characters, is represented in the previous

[1] The Latin writers of the Middle Ages and of the Renaissance spoke of "leading" a number "into" this manifoldedness (*multus*, many, *plicare*, to fold), as when Jordanus Nemorarius (c. 1225) says: "Si aliquis numerus . . . ducatur." So it is not unusual in algebra to hear the expression "a+b into a—b," by which the product of the two binomials is meant. See Smith, *Mathematics*, pp. 95, 96.

[2] "In Bezug auf die Multiplikation unterscheidet man auch dem Ausdrucke nach erst um die Mitte des 16. Jahrhunderts scharf '*signum, numerus, character*' des einzelnen Gliedes (Scheubel)," Die Coss, p. 38.

[3] "Exponentes signorum, in multiplicatione adde, in divisione subtrahe, tunc fit exponens signi fiendi." (Stifel *Arithmetica Integra*, fol. 236). Treutlein, *Die Coss*, p. 39.
Recorde, *Whetstone of witte* (1557), S 4, *v*, "By this table maie you easily knowe the signe (of the exponent) that shall serve for your newe somme in multiplication."



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table, then the characters of the products can be found from the table. This table can be extended, however, to a series of numbers immediately following.[1]

Four Examples of Multiplication

$8x$	7	$5x^3$	$13x^4$
5	7	$7x^2$	$8x^4$
$40x$	49	$35x^5$	$104x^8$

We can determine from the following rules whether the sign + or − is to be written in the product:

1. When the signs of the quantity to be multiplied (the multiplicand), and the quantity by which we multiply (the multiplier), are the same, the number which is produced through multiplication will have the affirmative[2] sign, +.

2. When the signs are different,[3] the number produced through multiplication will have the negative sign, −.

3. Suggestion or rule three. The reader is reminded that the beginning of the regular order of succession of numbers is to be represented by the sign plus, +.

Two Examples

$$\begin{array}{r} 7x^2 + 4x - 5 \\ 7x \\ \hline 49x^3 + 28x^2 - 35x \end{array}$$

$$\begin{array}{r} 7x^4 + 8 - 6x \\ 7x^3 - 4x^2 \\ \hline -28x^6 - 32x^2 + 24x^3 \\ 49x^7 + 56x^3 - 42x^4 \\ \hline 49x^7 - 28x^6 - 42x^4 - 32x^2 + 80x^3 \end{array}$$

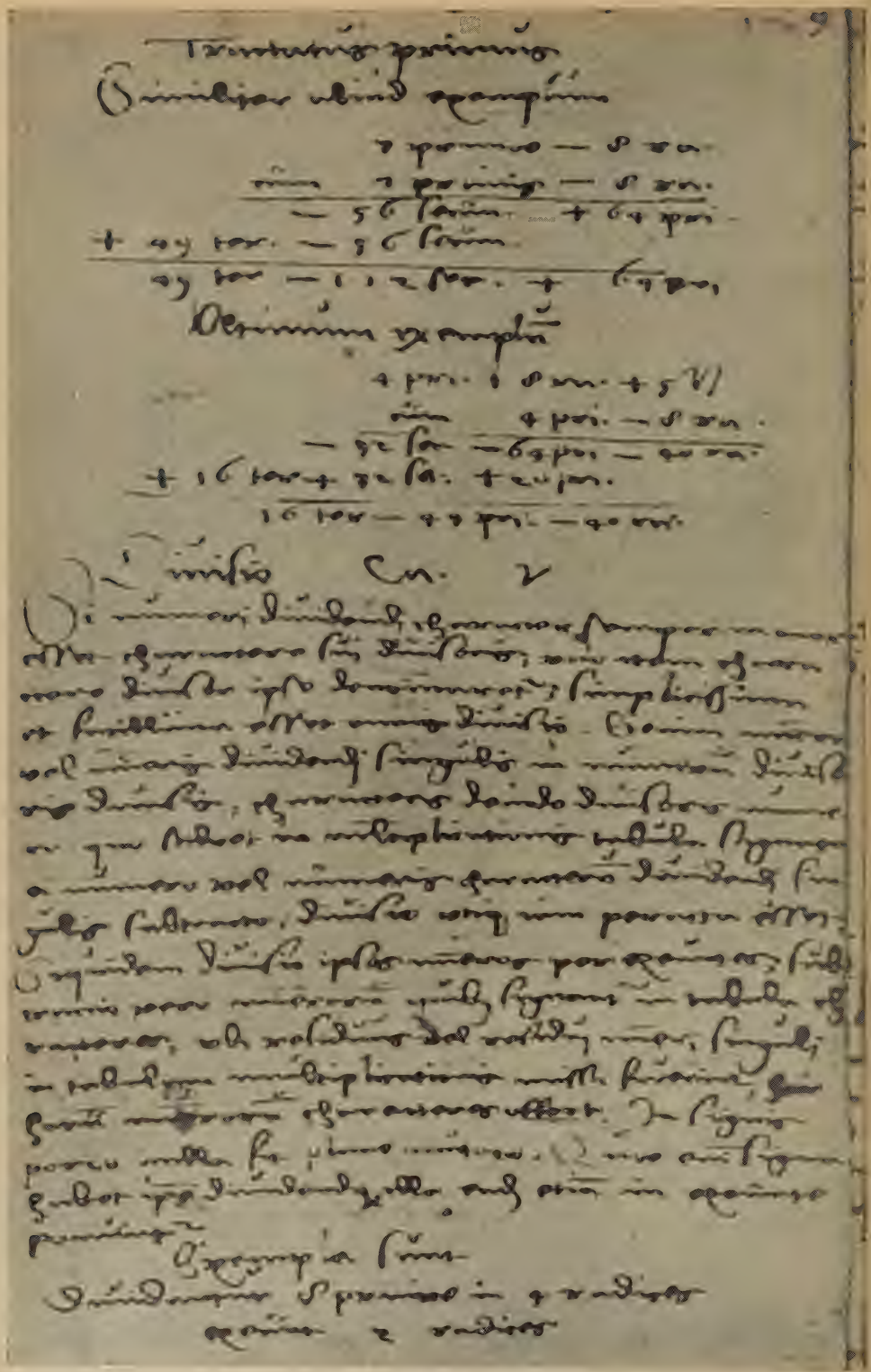
Another Example

$$\begin{array}{r} 9x^3 - 8x \\ 6x^2 \\ \hline 54x^5 - 48x^3 \end{array}$$

[1] On page 2, *v*, of Scheubel's 1551 edition, we find "Quia vero hae numerorum appellationes in infinitum sese extendunt."

[2] Tropfke, *Geschichte*, I, p. 168, "Scheubel used positive and negative numbers in his algebra which he published with the six books of Euclid (1550), 'signum affirmativum und privativum vel negativum.'"

[3] Scheubel, in his "Additions to Algebra" (p. 123–157 of this MS.), says "Positives by negatives you subtract, but other products you add." See Karpinski, *al-Khowârizmî*, p. 139. Brahmagupta (c. 628), in the algebraic part of his work, "the *pulverizer*," gives a rule for multiplication of signs involved in indeterminate equations. Diophantus employs this same rule for signs. (Heath, *Diophantus*, p. 130.) Recorde, *Whetstone of witte* (1557), fol. x 4, *r*, gives a shorter rule: "In multiplication, there is no difficultie so that you dooe well marke the signes + and −, which beyng bothe like, will have the signe + sette in the totalle and beyng unlike, thei will have in the totalle the signe −."



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Another Similar Example

$$\begin{array}{r} 7x^2 - 8x \\ 7x^2 - 8x \\ \hline -\ 56x^3 + 64x^2 \\ 49x^4 - 56x^3 \\ \hline 49x^4 - 112x^3 + 64x^2 \end{array}$$

The Last Example

$$\begin{array}{r} 4x^2 + 8x - 5 \\ 4x^2 - 8x \\ \hline -\ 32x^3 - 64x^2 - 40x \\ 16x^4 + 32x^3 + 20x^2 \\ \hline 16x^4 - 44x^2 - 40x \end{array}$$

Chapter V

Division[1]

If the character of the number of the dividend is always larger than the character of the divisor, and if the divisor is always designated specifically by one and the same character, (then) every division is very simple and easy. For when the number or numbers of the dividend have been divided singly by the number of the divisor and when the number of the character of the divisor, so far as it is indicated in the multiplication table, has been subtracted from the number or the individual numbers of the characters of the dividend, then the division is completed.

Since the characters of the numbers in division are to be subtracted in forming the quotient, we may refer to our multiplication table for their characters.[2] If it is clear that there is no change in signs, those signs which occur in the dividend will be placed in the quotient.

Examples

$$8x^2 \div 4x = 2x$$

[1] The late Latin writers thought of division as measuring. This idea originated from Euclid's (c. 300 B. C.) expression *metrein*, or parting, which Heron, Pappus, and Diophantus expressed by *merizein*. We find this double usage in the sixteenth century among such writers as Baker (1568), who speaks of "Deuision or partition," and Digges (1572), who uses the expression "To deuide or parte." See Smith, *Mathematics*, p. 97.

[2] In his 1551 edition, Scheubel gives the following rule at the end of his discussion of division: "Afferri autem huc necesse est tabulam, in multiplicatione, pro characteribus productorum habendis, expositam. Nam quemadmodum in multiplicatione, numeri characterum eorum qui interse multiplicantur, pro characteribus productorum habendis addendi, sic in divisione iam, ut habeatur character exeuntis unius, divisoris scilicet character de numero characteris ipsius dividendi subtrahi debet. Per residuum enim numerum statim, in tabula illa, exeuntis character manifestabitur: cum is nimirum sit, cui est, numerus residuus supra positus."

Regulæ in Algebra

Item si numero in se multiplicato producitur numerus, radix illius numeri est dimidium numeri.

Sic si quadratus in 10 numeros producit 40, radix quadrati est 4.

Quando aliqua æquatio in aliquod quadratum producitur, radix illius quadrati est dimidium illius æquationis.

Alia exempla

Si quadratum + 3 æquantur 4 radicibus, radix quadrati est 2 + 1.

Item si quadratum + 12 æquantur 6 numeris, radix quadrati est 3 + 1.

Sed quia non semper accidit, quod numerus, qui multiplicatus in se, producit numerum, ipse sit dimidium numeri, qui producitur, ideo, quia non semper, ubi numerus ipse multiplicatus in se

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$$\begin{aligned} 11x^4 \div 4x^4 &= 2\tfrac{3}{4} \\ 9x^3 \div 4 &= 2\tfrac{1}{4}x^3 \\ 7x^4 \div 3x^2 &= 2\tfrac{1}{3}x^2 \\ 7x^4 \div 8x^2 &= \tfrac{7}{8}x^2 \end{aligned}$$

Other Examples

$$\begin{aligned} (12x^2 + 9x) \div 3x &= 4x + 3 \\ (27x^2 - 12x) \div 6 &= 4\tfrac{1}{2}x^2 - 2x \end{aligned}$$

But it often happens that the power of the divisor is greater than that of the dividend, or more than one character is given; then, from the many characters, that number which is represented by a single character, and which comes first, is the dividend. Then the divisor should be placed under the dividend and the rule applied.

Thus, if we are to divide $12x^4$ by $4x^2 - 6x$ and also $18x^3 - 4x$ by $8x^2 + 6$, the quotients will be:

$$\frac{6x^3}{2x - 3} \text{ and } \frac{9x^3 - 2x}{4x^2 + 3}.$$

A TREATMENT OF FRACTIONS

Chapter I

Notation and Numeration of the Second Algorism

We have here the same notation of fractions which is ordinarily used in common fractions, except that characters are annexed as appropriate names.

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If any one of two characters is expressed with the numerator, then the other is expressed with the denominator; but if there is only one character in the fraction, we know that it belongs to the numerator of the fraction.[1]

As an example: $\frac{4}{9x}$ indicates 4 numbers divided by $9x$. In the same way $\frac{9x}{11}$ means $9x$ divided by 11. Similarly all fractions can be explained.

The general rule for addition is used in adding fractions.

Chapter II

Addition and Subtraction

The characters of the collected numerators and denominators of the fractions, or the sums of the fractions, are added from the table explained in the multiplication of whole numbers. The addition is then completed.

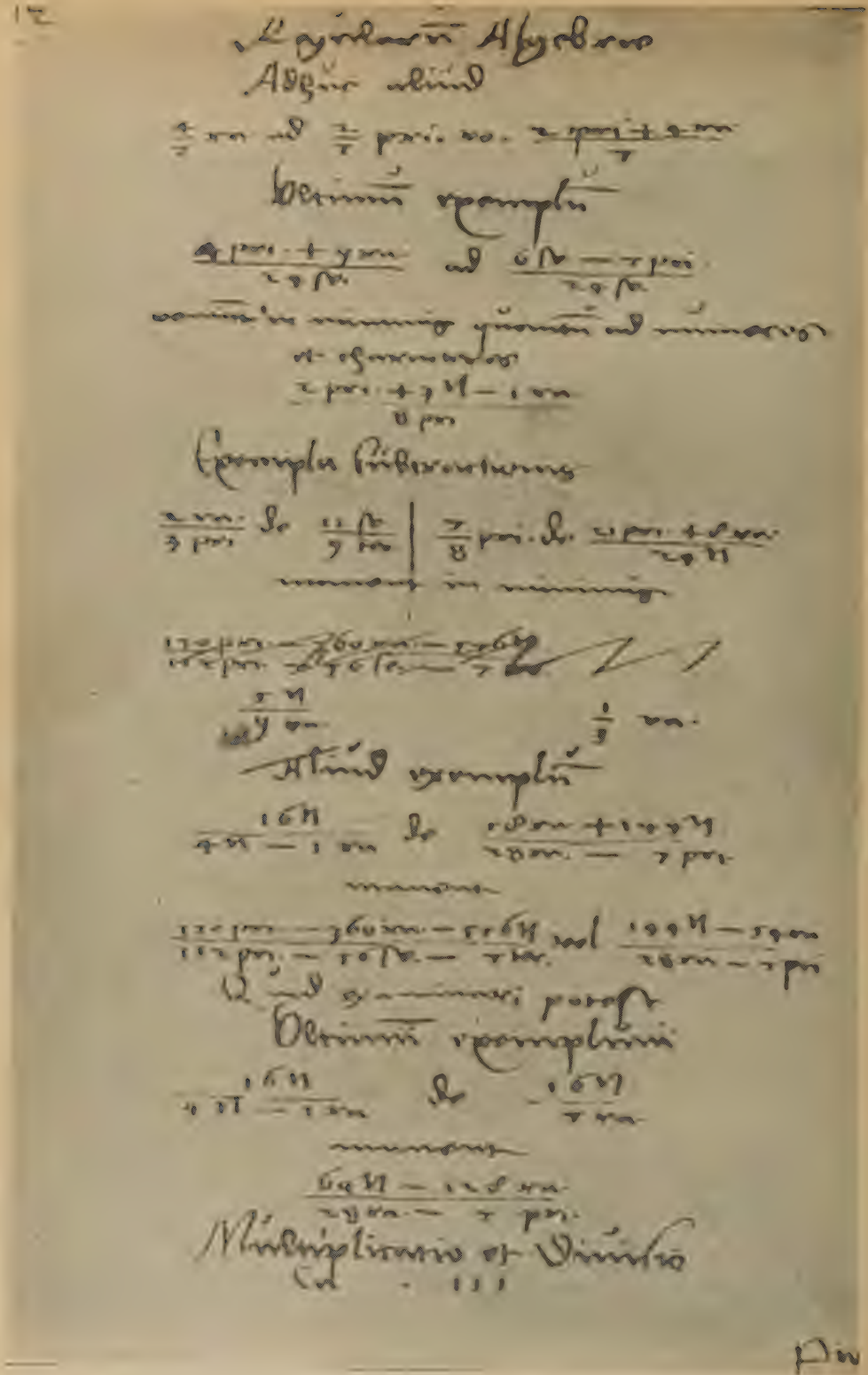
The general rule for subtraction is applicable to subtraction of fractions. The characters of the numerator and of the denominator of the remaining fraction are found from the table of whole numbers. The subtraction is then completed.

Examples of Addition

$$\frac{2}{3x^2} + \frac{5x^2}{7x^4} = \frac{29x^4}{21x^6} = \frac{29}{21x^2} \qquad \frac{2}{3x^2} + \frac{5x^2}{7x^3} = \frac{14x^3 + 15x^4}{15x^5}$$

$$\frac{12}{5x} + \frac{12}{6x^2 - 7x} = \frac{72x - 24}{30x^2 - 35x}$$

[1] Scheubel's *Algebra* (1551), just before his discussion of addition, has: "Sequuntur regulae quas observari in fractionibus oportet. Quantum ad enunciationem fractionum seu minutiarum, nulla est hic difficultas cum hae non aliter atque in communi minutiarum tractione enuntientur, nisi quod etiam vocabula seu characteres suis appellationibus efferantur, et horum quiden primus, si duo fuerint numeratorem: alter deinde, ipsum denominatorem sequatur."



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Another Example

$$\frac{4x}{7} + \frac{2x^2}{7} = \frac{2x^2 + 4x}{7}$$

A Last Example

$\frac{4x^2 + 9x}{24x^3} + \frac{6x^3 - 7x^2}{24x^3} = \frac{2x^2 + 3 - x}{8x^2}$, when reduced to lowest terms.

Examples of Subtraction

$$\frac{11x^3}{9x^4} - \frac{2x}{3x^2} = \frac{5}{9x} \qquad \frac{21x^2 + 8x}{24} - \frac{7x^2}{8} = \frac{1}{3}x, \left(\frac{x}{3}\right)$$

$$\frac{58x + 144}{20 - 7x^2} - \frac{16}{4 - x} = \frac{170x^2 - 360x - 576}{112x^2 - 56x^3 - 7x^4} \text{ or } \frac{144 - 54x}{28x - 7x^2},$$

which can be proved.

A Last Example

$$\frac{16}{7x} - \frac{16}{4 - x} = \frac{64 - 128x}{28x - 7x^2}$$

Chapter III

Multiplication and Division

Tractatus primus

Pro multiplicatione. multiplicare nil aliud est, quam ex duobus, pro quibus multiplicatio fit, unum producere, quod singulis illorum aequale sit, ut supra est explicatum, absolvitur: et multiplicatio absolvitur ope huius regulae in duobus, ubi unum tantum in alio primo, est autem duplex, multiplicatio aequalium et superiorum gradus positi, sive

Exempla multiplicationis

$\frac{3}{4}$ N in $\frac{8}{9}$ pri. | $\frac{9 \text{ qu.} + 6 \text{ m.}}{8 \text{ m.}}$ in $\frac{16 \text{ N}}{9 \text{ m.}}$

prodit $\frac{2 \text{ m.}}{3 \text{ pri.}}$ | prodit $\frac{6 \text{ pri.} + 4 \text{ N}}{3 \text{ m.}}$

Aliud exemplum

$\frac{8 \text{ m.} + 6 \text{ m.}}{7 \text{ pri.} - 3 \text{ qu.}}$ cum $\frac{14 \text{ pri.} - 3 \text{ N}}{8 \text{ m.} + 6 \text{ m.}}$

prodit $\frac{14 \text{ pri.} - 3 \text{ N.}}{7 \text{ pri.} - 3 \text{ qu.}}$

Quartum exemplum

$\frac{5 \text{ pri.}}{8 \text{ lat.}}$ in 6 m. — 11. prod. $\frac{15 \text{ m.} - 40 \text{ N}}{4 \text{ pri.}}$

Ultimum exemplum

$\frac{4 \text{ m.} + 6 \text{ N}}{9 \text{ pri.} - 12 \text{ N}}$ cum $\frac{7 \text{ pri.}}{8 \text{ m.}} - 5 \text{ N}$

prodit

$\frac{\text{cub.} - 11 \text{ qu. pri.} - 240 \text{ m.}}{72 \text{ qu.} - 96 \text{ m.}}$

Exem-

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The numerators and denominators should be multiplied together according to the general rule for multiplication. When appropriate characters are taken from the table and are annexed to represent the powers of the single products, the multiplication is completed. Similarly in division,[1] when first one number is divided by another according to the rule (of division), then the characters of the quotient should be found from the above (table).

Examples of Multiplication

$$\frac{3}{4}\left(\frac{8x}{9x^2}\right) = \frac{2x}{3x^2}$$

$$\left(\frac{9x^3 + 6x}{8x}\right)\frac{16}{9x} = \frac{6x^2 + 4}{3x}$$

$$\left(\frac{8x^4 + 6x}{7x^2 - 3x}\right)\left(\frac{14x^2 - 3}{8x^4 + 6x}\right) = \frac{14x^2 - 3}{7x^2 - 3x^3}$$

A Fourth Example

$$\frac{5x^2}{8x^4}(6x - 4) = \frac{15x - 10}{4x^2}^{2}$$

The Last Example

$$\left(\frac{4x + 6}{9x^2 - 12}\right)\left(\frac{7x^2}{8x} - 5\right) = \frac{28x^3 - 118x^2 - 240x}{72x^2 - 96x}$$

[1] Brahmagupta (c. 628), in "the *pulverizer*" gives the first printed rule for division of fractions: Invert the divisor and multiply.

Scheubel has $\frac{15x - 40}{4x^2}$.

Regulæ Algebræ

Exempla diuisionis

Diuidantur

$\frac{3xx}{+px}$ in $\frac{8qxx}{8xx}$ erit $\frac{2px}{c xx}$ in $\frac{3xx}{+px}$

Exeunt $\frac{2N}{3px}$ Exeunt $\frac{3px}{2N}$

Aliud exemplum

$$\frac{6px + 15N}{15xx} \text{ in } \frac{cxx + 6N}{10xx}$$

Exeunt in minimis

$$\frac{2px + 5N}{+xx - 3N}$$

Regula proportionum

Regula de proportionibus cum ea vtuntur quæ in numeris vsitatissima, facilem descriptionem, nempe quod, positis vtrinque cum suis signis ordine primis, secundis, vtrinque in tertio multiplicatis, atque vtrique inde producto in primum secundæ, ratio vtriusque rem multiplicatæ ab eadem diuisoris vtrinque ablatæ, proxime exemplis nihil parandæ diligentiæ.

Exempla vnius rei

Primum $+x$ vnc. valent $+4N$

quinq: $+5$ primæ

fiunt $+16$ vnc.

Secunda $+29$ vnc. valent $+9px + 6N$

quinq: $+5$ secundæ $-6xx$

fiunt $+45xx - 24px - 36N$

$+29N$

Quod examinari potest

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Examples of Division

$$\frac{3x}{4x^2} \div \frac{9x^2}{8x} = \frac{2}{3x^2} \qquad \frac{9x^2}{8x} \div \frac{3x}{4x^2} = \frac{3x^2}{2}$$

$$\frac{6x^2+15}{15x} \div \frac{8x-6}{10x} = \frac{2x^2+5}{4x-3}$$

The Rule of Proportion[1]

Since the rule of proportion is not different from the rule used in common arithmetic, which rule includes the arrangement of characters in order, the multiplication of the second number by the third, and the division of this product by the first number,[2] and since the rules for multiplication and division have been thoroughly taught and described above, only the following examples are given.

First Example

$$7x : 14 = 8x^2 : ?$$
$$7x : 14 = 8x^2 : 16x$$

Second Example

$$29x : 9x^2 + 6 = 5x^3 - 6x : ?$$
$$29x : 9x^2 + 6 = 5x^3 - 6x : \frac{45x^4 - 24x^2 - 36}{29},$$

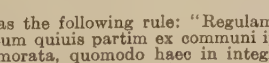
which can be proved.

[1] Cajori, *A History of Mathematics*, New York, 1919, p. 16, says: "The Ahmes papyrus actually shows that the rudiments of proportion were known to the Egyptians;" hereafter referred to as Cajori, *History*.

Euclid, *Elements*, Book VII, contains the theory of proportion of (rational) numbers developed on the basis of the definition, "Numbers are proportional when the first is the same multiple part or parts of the second that the third is of the fourth." This is considered the older Pythagorean theory of proportion.

Jordanus Nemorarius (c. 1225), in his *De Numeris Datis*, problem 30, says, "If there should be four numbers in proportion and three of them should be given, the fourth would be evident."

[2] Recorde, *Whetstone of witte* (1557), fol. C c 3, *v*, gives the following explanation: "Of the Golden Rule, or rule of proportion, should followe now, by the common order. But seyng there is no difficultie in it, nother any other forme of woorke, then is in vulgare nombers, I will not staie any tyme aboute it."



Scheubel's 1551 edition has the following rule: "Regulam de proportionibus, quae nunc recto ordine sequi deberet, cum quiuis partim ex communi ipsius descriptione, partim ex iis quae hactenus sunt commemorata, quomodo haec in integris atque etiam in fractionibus tractari debeat, facile cognoscat: Lectori satis me facturum, uno duntaxat atque altero exemplo sum opinatus."

Tertium. Quinque prima — 2 radicibus
valent 12 tertios — 6 N. quinta 6 pri — 3 ra
— fiunt 3 6 quin — 18 ter — 18 ra + 2
2 ra. — 1 N.

Vel quinta 6 prima — 3 radicibus
fiunt 4 pri. — 4 ra. + 2 N
2 ter — 1 N.

Similiter modo in fractoribus reperies.
Sic proponas posthac

$\frac{2}{3}$ N. valent $\frac{5}{8}$ ra. quinta $\frac{1}{4}$ pri.
fiunt $\frac{5}{8}$ ter.

Item $\frac{4}{5}$ pri. valent $\frac{8}{15}$ ra. quintam $\frac{2}{9}$
fiunt $\frac{2}{6}$ N.
" " $\frac{6}{9}$ pri.

Reperiuntur autem Algorismi de
fractis quorum declinatio pri-
mo tendenda est.

Numeri rupti sunt, quorum radices
quae declarantur, certo numero [illegible]
rari, vocabulo appellantur. Ut [illegible]
[illegible] Ad eos [illegible]
[illegible] appellatur, [illegible]
[illegible] vel [illegible] [illegible]
[illegible] [illegible] [illegible]
[illegible] [illegible] [illegible]
[illegible]

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$$4x^2 - 2x : 12x^4 - 6 = 6x^2 - 3x : ?$$

$$4x^2 - 2x : 12x^4 - 6 = 6x^2 - 3x : \frac{(36x^5 - 18x^4 - 18x + 9)}{2x - 1}$$

$$12x^4 - 6 : 4x^2 - 2x = 6x^2 - 3x : ?$$

$$12x^4 - 6 : 4x^2 - 2x = 6x^2 - 3x : \frac{(4x^2 - 4x + 1)}{2x^4 - 1}\, x^2$$

The following illustrations show that the same method is used in fractions:

$$\frac{2}{3} : \frac{5x}{6} = \frac{1}{4}\, x^2 : ?$$

$$\frac{2}{3} : \frac{5x}{6} = \frac{x^2}{4} : \frac{5}{16}\, x^3 \qquad [1]$$

$$\frac{4x^2}{5} : \frac{8x}{15} = \frac{7}{9x} : ?$$

$$\frac{4x^2}{5} : \frac{8x}{15} = \frac{7}{9x} : \frac{14}{27x^2} \qquad [2]$$

The Algorism of Surds

DEFINITION OF SURDS

Surd numbers[3] are numbers whose desired roots cannot be signified or expressed by a certain (rational) number. Although we may consider a number entirely rational, whenever the root which is sought cannot be expressed by a certain number, it is called irrational and a surd. Similarly, 21, 15, 19, 347, 8700—also many

[1] Scheubel had $\frac{5}{8}\, x^3$

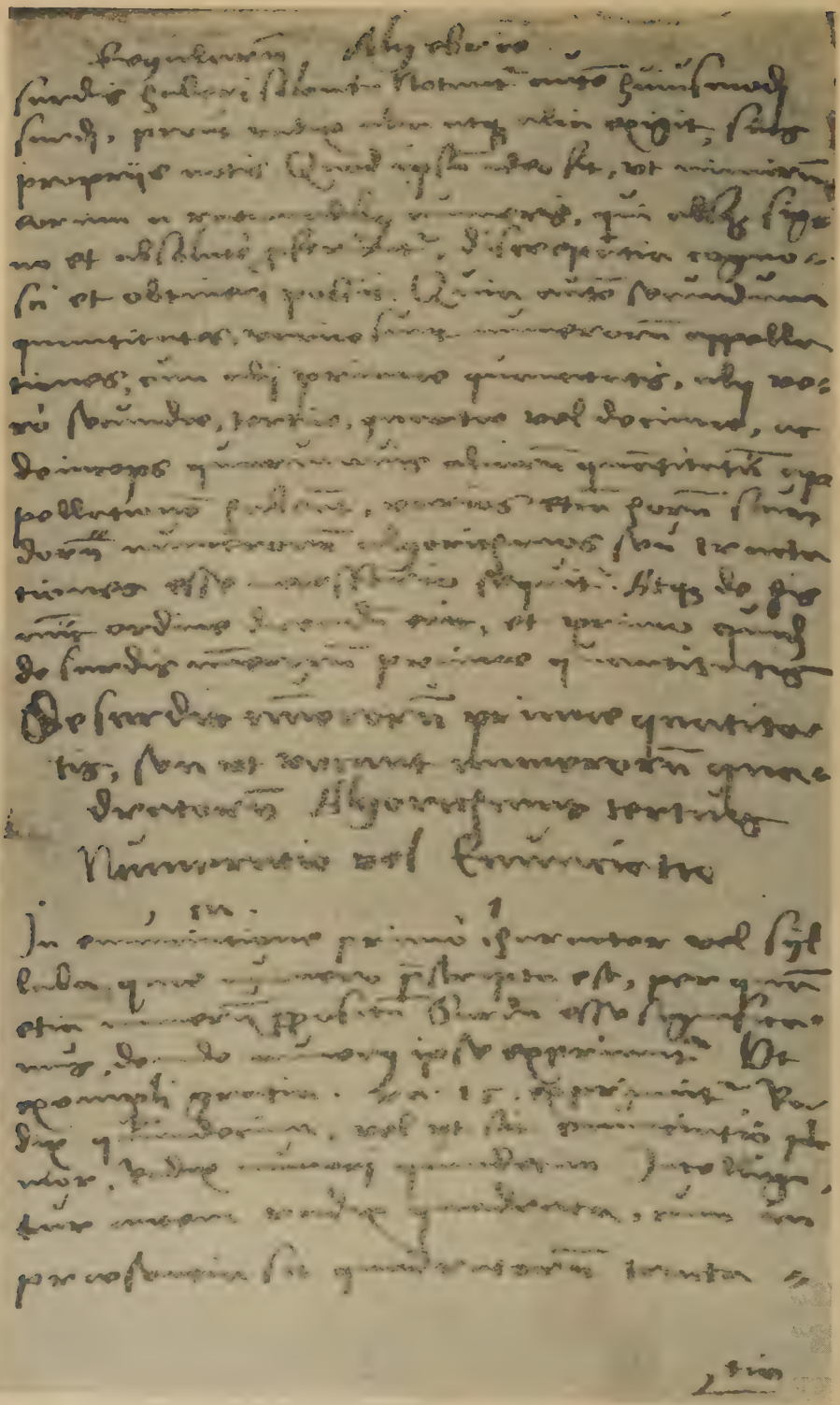
[2] Scheubel had $\frac{7}{6x^2}$

[3] The term "surd" is a translation from the Euclidean term *a'logos* (without ratio, irrational, incommensurable). There seems to be good reason for believing that the immediate followers of Pythagoras knew and demonstrated the incommensurability of the diagonal and the side of a square. But they looked on this case of irrationality as a peculiarity of the square, Smith, *History*, II, p. 251.

So far as is now known the first European use of the word is by Gherardo of Cremona (c. 1150). Fibonacci, in his *Liber Abaci*, refers to the name to represent a number without a root. Boncompagni, *Scritti*, I, p. 353.

Treutlein, *Das Rechnen im 16. Jahrhundert*, p. 67, "Und schon vor Scheubel findet (1540), bezeichnet es deutlicher als Annäherungsverfahren und betont die Unmöglichkeit der Auffindung der Wurzel aus einem *numerus surdus*, . . . was wir so bei Scheubel und Gemma vermissen. . . ."

Robert Recorde, *Whetstone of witte* (1557), fol. s 1, *r*, "Nombers *radicalle*, which commonly bee called nombers *irrationalle*: bicause many of them are soche, as can not bee expressed, by common nombers *abstracte*, nother by any certain rationalle nomber. Other men call them more aptly *surde nombers* (Def. 5th Euclide 10th)"; fol. L 1 3, *r*, A *surde nomber* is nothyng els, but soche a nomber set for a roote, as can not be expressed by any other nomber absolute, as $\sqrt{10}$ or $\sqrt{8}$ or any nomber, that is not a square."



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other numbers are usually considered as surds. The name applied to surds depends upon the kind of root which is to be extracted. Because there are, however, various names given to numbers according to their manifold powers, and since all of these (numbers) have names such as the first (quantity), the second (quantity), the third (quantity), the fourth (quantity), and so on up to the tenth (quantity) successively in the order of the power of all these quantities, it follows necessarily that there is a varied algorithm or notation of these surd numbers. Since (surd numbers) follow in succession, (surds) of the first quantity are to be treated first.

CONCERNING ROOTS OF NUMBERS OF THE FIRST QUANTITY, OR THE THIRD ALGORISM

QUADRATIC NUMBERS

Chapter I

Numeration or Notation

In the first notation a number is expressed as a surd[1] by attaching a character or a syllable to it. The root of 15 is expressed by ra. 15, or more definitely, the (square) root of 15, ($\sqrt{15}$). Since square root is to be explained in the following treatment under the discussion of quadratics, it is necessary to consider next cube

[1] Surds, formerly belonging to the theory of numbers, the ancient *arithmetica*, were first included in algebra in the Renaissance. Pacioli's *Suma* (1494), has a discussion of surds in arithmetic just before his discussion of algebra.
Recorde, *Whetstone of witte* (1557), fol. L I 3, *v*, says: "Howbeit many times it hath with it, for the moare certaintie the *Cossike* sign. 3—√3.20, the square roote of 20."

Tractatus primus

[illegible] Multiplicatio

Multiplicatio [illegible] est [illegible]

[illegible]

Exempla [illegible]

[illegible]	[illegible]
prod. [illegible]	prod. 12
Alia [illegible]	exempla

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root, or root of the second quantity. A general common sense explanation of the representation or notation of the root by Ra. is necessary. Many have successfully designated these roots with a point,[1] or with a line erected, extending from the right side.[2] Thus when the square root of any number is desired, place before it this notation ($\surd$). Cube root and fourth root (root of the root) will be explained to the reader later.

Chapter II

Multiplication

Multiplication of surds in general is the contraction of the roots of surd numbers under the root of another surd (number) in such a way that the product of the units will be indicated.[3] This is done by multiplying the rational part of one number by the rational part of the other number, neglecting the character. When the root of this product is taken, the result is the multiplication of surd numbers.

Examples

$$\surd 6 \cdot \surd 3 = \surd 18 \qquad \surd 18 \cdot \surd 8 = \surd 144 = 12$$

Two Other Examples

[1] "Schon in der Mitte des 15. Jahrhunderts ein besonderes Zeichen für die Quadratwurzel gebraucnt wird, beweist das mehrfact erwähnte Wiener Manuscript. . . . man solle beim Quadriren einer Quadratwurzel den vor der betreffenden Zahl stehenden 'Punkt' einfach weglassen . . . welches beim Schreiben dann wohl in einen 'Punkt' mit angehängtem Strichlein überging. Während dann Riese nur das einfache Zeichen $\surd$ für die zweite Wurzel gebraucht, bemerken wir, dass Rudolff 'vermerckt von kürtz wegen radix quadrata mit solchem character $\surd$. Als $\surd$ 4, bedeutet radicem quadratam ausz 4 ist 2,'" Treutlein, *Die Coss*, p. 45.

[2] In papyri found at Kahun we have our first quadratic equations with a symbol used to designate square root, Cantor, I (3), 95–96.

Stevin (Arithmetique 1585, Girard edition, 1634, pp. 10, 19) uses Rudolff's symbols, but he uses $\surd$(3) for cube root and $\surd\surd$(3) for the fourth root of the cube root.

[3] Scheubel, in his "Additions to Algebra" (pp. 123–157 of this MS.), says: "If you wish to multiply the root of the number 9 by the root of the number 4, multiply 9 by 4 and the root of the product is that which you seek," Karpinski, *al-Khowârizmî*, p. 143.

Regulæ Algebræ

$\sqrt{5\frac{1}{3}}$ cum $5\frac{1}{3}$ | $14\frac{3}{3}$ cum $\sqrt{21}$

prov. ra. $44\frac{2}{9}$ | prov. $215\frac{1}{9}$

In qua solum exemplo, quia radix numeri surdi fuit, alter radicis valor, ab illo numero subtractus, numerum rationalem dedit, sed cum surdo quadrato nihil multiplicatum in solutione fuisse apparet. Etsi in surdorum tractatione, tum quia in regula ejus in fractionibus, semper error appellationis vitandus est, necesse est. Regula hæc, quod cum surdis dolosa est appellatio: Duplatum quinque fuerit cum 4, fac est cum binis quadratis: duplatum vero ut quadratum duplatum et si quis alius præter ea fabricam multiplicaverit, cum illorum multitudine quadrilatera, libri vi. 16. ex eadem demonstratione, patebit etc. ut supra.

Regulæ exemplo

$\sqrt{6}$ bis $\sqrt{6}$ ter $\sqrt{6}$ quater

sunt $\sqrt{24}$ $\sqrt{54}$ $\sqrt{96}$

Divisio m. III

Divisio surdorum in genere. est in numeris numeri, cum radix vel fabricatus, quotus radix dividendo cuncta est in ipsa radice, dividenda. Hanc ita practicus divisores cum numeris omnibus in numeris rationalibus vitam.

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$\sqrt{8\ \frac{1}{3}} \cdot \sqrt{5\ \frac{1}{3}} = \sqrt{\frac{25}{3} \cdot \frac{16}{3}} = \sqrt{44\ \frac{4}{9}}$; $14\frac{2}{3} \cdot \sqrt{21} =$

$\sqrt{215\ \frac{1}{9}} \cdot \sqrt{21}$[1]

In this last example, where one number is a surd and the other one is a rational number, multiplication is performed, not with the rational number, but with its square. In the treatment of surds, this rule is to be followed because it is necessary for all numbers to be of the same denomination. Since the name (of the number) should be the same as the surd, duplication (or squaring) is involved as in the case of 4, which is the square of 2, and similarly, in succession other squares (are involved), such as, 9, 16, 25 . . . ; thus triplication, quadruplication, and many other multiplications can be used.[2]

Examples Follow

$\sqrt{6} \cdot 2$, $\sqrt{6} \cdot 3$, $\sqrt{6} \cdot 4$ are, respectively: $\sqrt{24}$, $\sqrt{54}$, $\sqrt{96}$.

Chapter III

Division

Division[3] of surds in general is the finding of a number whose root has as many units as the dividing root is contained in the root which is to be divided. This is found by the division of one rational number by another rational number, neglecting the char-

[1] Scheubel omitted $\sqrt{21}$.

[2] Scheubel, in his "Additions to Algebra" (pp. 123–157 of this MS.), concerning the doubling, tripling and quadrupling of radicals, says: "When you wish to double the root of a square, either a definite root or a surd, you multiply 2 by 2 and multiply the given square by this product. The root of this product will be the double which you seek. Similarly if you wish its triple you multiply 3 by 3. And if you wish the quadruple of it, you multiply 4 by 4, and so on." See also Karpinski, *al-Khowarizmi*, p. 141.

[3] In Scheubel's "Addition of Algebra" (pp. 123–157 of this MS.) concerning division, we find: "Notice, if you wish to divide the root of 9 by the root of 4, divide 9 by 4, giving 2–. Of the result take the root, which is $1\frac{1}{2}$, and the resulting number will be the quotient of the root divided by the root. But if first you wish to divide two roots of 9 by the root of the number 4, you seek first the double of the root of 9 according to that which we have explained, and divide the product by the number 4." See also Karpinski, *al-Khowarizmi*, p. 143.

Recorde, *Whetstone of witte* (1557), fol. Mm 1, *v*, "Reduction of surdes is the reduction of surdes of sundrie denomination into one denomination. So if you would double $\sqrt{8}$, that is to say, if you would multiply it by 2 you must first multiply that 2 squarely, and then multiply those two numbers together. That is to saie you shall multiply $\sqrt{3}$ 8 by 3. 4 and so it is doubled. Likewise to triple any square surde is to multiply it by 9, to quadruple it is to multiply it by 16 and so forthe."

Exemplum

ra. 42 in ra. 6 | √48 in √3
erit √7 | erit 4

Alia exempla

√8 3/4 in √7/9 | 3/4 in √2/3
erit √1/5 | erit √ 11/17

Similiter ra. 24 in
2. erit √6. 3. erit √2 2/3. 4. erit √1 1/2

Et quod de pluribus proponitur.

Additio ra. 1111

Additio surdorum in genere, est radicum proximarum surdorum in unam summam collectio. Hoc modo fiet ea. Additi quorundam surdorum, quando eorum surdi nihil aliud, nisi alterum multiplorum, quod inde producit duplorum, postea duplum id quod ex aggregatis additionibus: et erit quod collecti radicis quorundam radicum surdorum proximarum summa. Cuius igitur radicis quorumcunque nunc additorum, additio peracta erit

Exemplum

Additio ra. 8 ad ra. 12

Quorundam surdorum summa | ra. 20
horum surdi ut alterum multi: | √96
Bis. Summa | √384
Quorundam rationum et duplum quod
ex summa | 20 et √384

Atq;

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acter, and when the root of the quotient is taken the result is the division of one surd number by another.

Examples

$$\sqrt{42} \div \sqrt{6} = \sqrt{7} \qquad \sqrt{48} \div \sqrt{3} = 4$$

Other Examples

$$\sqrt{8\tfrac{3}{4}} \div \sqrt{\tfrac{7}{8}} = \tfrac{1}{5}$$

$$\tfrac{3}{4} \div \sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{27}{32}}\,[1]$$

$$\sqrt{24} \div 2 = \sqrt{24} \div \sqrt{4} = \sqrt{6}$$

$$\sqrt{24} \div 3 = \sqrt{24} \div \sqrt{9} = \sqrt{2\tfrac{2}{3}}$$

$$\sqrt{24} \div 4 = \sqrt{24} \div \sqrt{16} = \sqrt{\tfrac{3}{2}}$$

Chapter IV

Addition

Addition of surds in general is the collection of the roots of the proposed surds into one sum. This is done in the following way:[2] The square of the surds are added, one surd number is multiplied by the other, then the double and the double of this is added to the sum of the squares, and the square root of this sum will give the sum of the roots of the proposed surds.

Example

$$\sqrt{8} + \sqrt{12} = 20 + \sqrt{384}$$

Sum of the surds	20
Multiply one surd by another	96
Doubling, doubling	384
The sum of the parts,	$20 + \sqrt{384}$.

[1] Scheubel has $\sqrt{1\ \tfrac{1}{16}}$, or $\sqrt{\tfrac{27}{16}}$

[2] Recorde, *Whetstone of witte*, fol. Mm 3, *v*, uses the same method as Scheubel:

"$\sqrt{26} + \sqrt{12}$ $\sqrt{38} + \sqrt{1248}$

$\sqrt{38}$ | $\sqrt{26}$
$\sqrt{12}$
$\sqrt{312}$
$\sqrt{4}$
$\sqrt{1248}$

Add in plaine forme their nombers together—putting the signe of the root and keep that as a part of the addition. Multiply the first two nombers together and their total multiplied by 4 and before that set the sign of the roote and it shall stand as the seconde part of that addition. Those 2 partes are added together by +."

Scheubel, in his 1551 edition, remarks: "Haec autem ex propositione 7 secundi Euclidis, perficitur hoc modo."

Regulæ Algebræ

Regula radicum quadratarum

Radix radicis 20 + √3 84

propositarum quadratum summa radicum erit.

Additionem signi radicis radicum quadratum radicis radicum radicis plus, vel per signum +. quod idem est, sic

ra. 12 plus ra. 8. Val √12 + √8

Compendium

Quia si radicis radicis radicum multiplicatio, producti radicis assignari queat, ut hoc illa producti radicis loco illius producti assumenda, ita in divisione simpliciter est. Quo facto, et brevior et expeditior erit operatio

Alia et brevior via additionis radicum quadratarum quæ radicibus quibusvis applicari possit

Reducantur primo signi radicis radicum ad numeros quadratos, quadratum radicis radicis simul addendi; et quod reliquum, signi quadrati cum radicibus quadratis radicibus multiplicetur: et erit signi producti radicis quadrata, radicibus quadratis applicanda. Quæ igitur radicis quadratæ aggregata, addenda, addito producto erit. Id quod per

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The square root of this sum or the root of $20 + \sqrt{384}$ is the sum of the roots of the proposed surds. But the roots of surd numbers are added more conveniently by the (word) plus, or by the sign $+$; thus,

$$\sqrt{12} \text{ plus } \sqrt{8}, \text{ or } \sqrt{12} + \sqrt{8}.$$

Compendium[1]

When one surd is multiplied by another, if the root of the product can be assigned, then the root of this product should be extracted and doubled. This is a shorter and more convenient process.

Another short method for the addition of surds which are considered commensurable is as follows:

First reduce the surds, if they are commensurable,[2] to square numbers, add the roots of the squares, and then multiply the square of this sum by the common measure of the commensurable surds. The square root of this product will be the sum of the roots of the proposed surds. Wnen that root is found, the addition is completed. This can be understood from the following examples:

[1] Compare with the 1551 edition: "*Appendicis Compendium.*
Habet haec operatio suum quoque compendium, in exemplis nimirum, ubi aliqua est in appellationibus numerorum convenientia and similitudo. Ut si, exempli gratia, hi duo surdi, ra 6 et ra ra 12, unus cum altero multiplicari, vel in alterum dividi debeat, numerus 6 quadrate tantum multiplicari, 12 vero prout sunt, ita absque immutatione relinqui debent."

[2] The geometric treatment of incommensurables naturally led to the arithmetic and algebraic treatment of irrationals. Euclid found the greatest common measure of two lines in his general development of commensurable magnitudes (Elements X, 3 and 4).

Exemplum premissum regulam radicis reperiri docebit

rad. 4 et rad. 6

Communis summarum numerus 6

Numeri quadrati 4 et 1

Radices 2 et 1

Radices simul collatae 3. in 1b,

Radicis summa rad. 54

Similiter alia exempla omnibus modis proponi, resolvi, probarique possunt per praecepta et theoremata data et patebit facilis probatio per se.

Exempla

$\sqrt{8\frac{1}{3}}$ et $\sqrt{6\frac{3}{4}}$ | $\sqrt{40}$ et $\sqrt{50\frac{5}{8}}$

summa $\sqrt{30\frac{1}{12}}$ | summa $\sqrt{180\frac{5}{8}}$

Decimum exemplum

Radix numeri 12 ad 4

Radicis radicis collecti

rad. 60 + 4 rad 4 + $\sqrt{12}$

Substantiae

Substantiae simplices in genere, est radicis numeri proposita simplicis ad numerum absolutum. Et ex his generibus quot modis diversae quadratae simplices, totidem simplices modi diversi, quod indicat quadratum

Digitized by

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ra. 24 and ra. 6 or $\sqrt{24} + \sqrt{6}$.
The common surd number 6.
The squares 4 and 1.
Roots 2 and 1.
Sum of the roots 3.
Square of this 9.
Common surd measure 6.
Multiply $\overline{54.}$
$\sqrt{54}$ is the sum of the roots.

In a similar way all other examples are solved, whether they are expressed by whole numbers, or by fractions, or by whole numbers and fractions.

Examples

$$\sqrt{8\tfrac{1}{3}} + \sqrt{6\tfrac{3}{4}} = \sqrt{30\tfrac{1}{12}}$$

$$\sqrt{40} + \sqrt{50\tfrac{5}{8}} = \sqrt{180\tfrac{5}{8}}$$

A Last Example

$$\sqrt{12} + 4 = \sqrt{\sqrt{768} + 28} \text{ or } 4 + \sqrt{12}.$$

Chapter V

Subtraction

Subtraction of surds in general is the subtraction of the root of one proposed surd from the root of the other. This method may be used. From the sum of the squares of the surds subtract twice the product of the surds; the square root of this remainder is called the root of the residual surd.

Regula vera Algebr.

Duplex est, posita scil. singulum a quadrato-
rum aggregato subducto vel: ut eius lx
radicis quadratum, radicis summi
radicis. Quadratum ipsius radicis quaesita
additum. Subtractio parenta erit.

Exemplum primum

√8 et √12	√12 et √50
manet radix rec.	ma. radix rec:
sibi 20 — √384	62 — √2400

Vel primum ad aequationem et
inventum binomium, manet

in priori quidem	in posteriore vero
√12 — √8	√50 — √12

Subtractum primum quidem quadratum ma-
net radix quaesiti pro parti-
bus illa Minus, vel per signum
— . quod dem rgl, sit

√12 minus √8 item √50 — √12

Alia et brevior via subtractionis
rerum quadratum qui in unam radi-
cem
duo dividere

Subtractus quadratis rerum quaesitarum ab
unde sit pro additis, nisi quod pro ad-
dito in radice subtracta, illum vero
rem altera addenda. Si deinde qua-
drata radicis residui rei, ut in additione
quadrata aggregati ac radicis, in altera
pro subtracto. Hoc facit quidem ad quae-
radicis summi, multiplicatur, et ac hoc pro-

az deinde

Finis

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When the root is found or indicated, the subtraction is completed.[1]

Examples

$\sqrt{8}$ from $\sqrt{12}$ = the residual root, $\sqrt{(20 - \sqrt{384})}$

$\sqrt{12}$ from $\sqrt{50}$ = the residual root, $\sqrt{(62 - 2400)}$

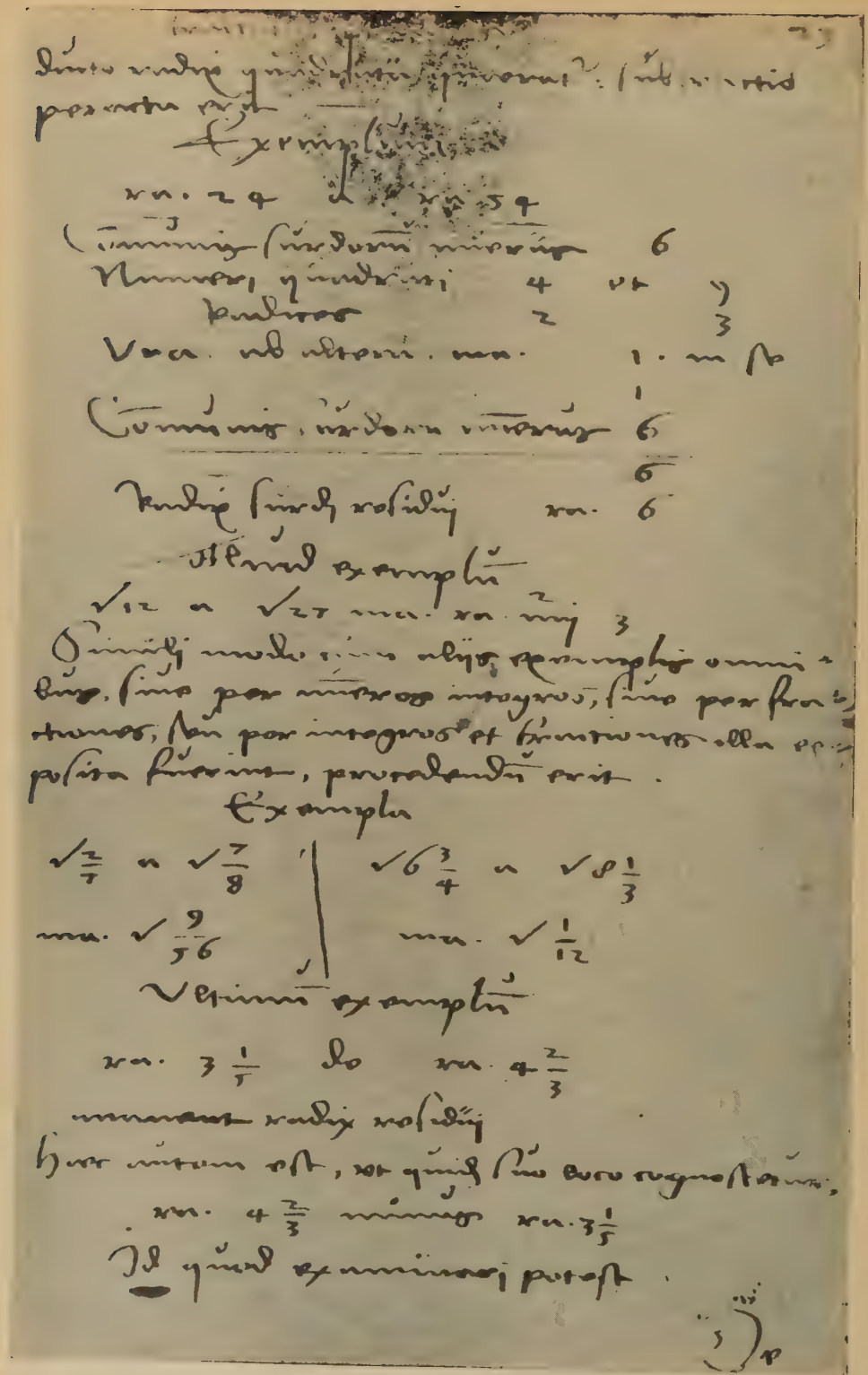
The following notation is more easily understood: $\sqrt{12} - \sqrt{8}$, the result of the first; $\sqrt{50} - \sqrt{12}$, the result of the second.

Roots of surd numbers of this type are more conveniently subtracted by means of the (word) minus, or by means of the sign $-$; thus $\sqrt{12}$ minus $\sqrt{8}$; or $\sqrt{50} - \sqrt{12}$.

ANOTHER SHORT METHOD FOR THE SUBTRACTION OF COMMENSURABLE SURDS

The method used for addition of commensurable surds can also be used here except that the roots are to be subtracted instead of added. If the square of the residual root is multiplied by the

[1] Scheubel's edition of 1551 gives the following reference: "Haec autem ex propositione 7 secundi Euclidis, perficitur hoc modo. Sumantur quadrata amborum, hoc est, eius a quo debet fieri substractio, ut totius, atque etiam radicis subtrahendae, ut unius partis lineae, vel numeri divisi."



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common number of the surd and the square root of this product is taken, the subtraction is completed.

Example

$\sqrt{24}$ from $\sqrt{54}$

The common surd number	6.
Squares	4 and 9.
Roots	2 and 3.
One from the other	1.
Square of this	1.
Common surd number	6.
Multiply	$\overline{6.}$

$\sqrt{6}$ = remainder of the roots.

Another Example

$\sqrt{12}$ from $\sqrt{27} = \sqrt{3.}$

The method of solution is the same for all examples, whether they are expressed in whole numbers, or in fractions or in whole numbers and fractions.

Examples

$\sqrt{\frac{2}{7}}$ from $\sqrt{\frac{7}{8}} = \sqrt{\frac{9}{56}}$

$\sqrt{8\frac{1}{3}} - \sqrt{6\frac{3}{4}} = \sqrt{\frac{1}{12}}$

A Last Example

Take $\sqrt{3\frac{1}{5}}$ from $\sqrt{4\frac{2}{3}}$

$= \sqrt{4\frac{2}{3}} - \sqrt{3\frac{1}{5}}$,

which can be proved.

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CONCERNING SURDS OF THE SECOND QUANTITY, OR SURDS OF CUBES[1]

THE FOURTH ALGORISM

Chapter I

Notation or Numeration

Since the numeration of the former third algorism concerning quadratic surds is clear, we have introduced here only the type of examples which will be sufficiently representative of such signs as ra. sec. 24, cube root, or root of the second quantity of 24. Ra. 21 is $\sqrt{21}$ (or an example of square root), but, since only cube root is to be treated here, we must proceed with other examples.

Frequently the syllable cu. is added to the syllable ra. to avoid confusion; since it is written as an addition to our former usage, such numbers as the cube root of 11 (ra. cu. 11, $\sqrt[3]{11}$), or the cube root of 13 (ra. cu. 13, $\sqrt[3]{13}$), and the cube roots of other numbers can be expressed.

Chapter II

Multiplication and Division

Multiplication and division are to be treated here more fully than they were in the development of quadratic surds; in the consideration of square root, results from the product of multiplication and the quotient of division were found, but in this treat-

[1] Euclid uses *cubus* for cube and Heron (c. 50?) does the same. Euclid's *Elements*, XI, def. 25. By the seventh century the cubic equation had begun to attract attention, as is evident from the *Ch' i-ku Sūanking* of Wang Hs' iao-t'ung (c. 625).

Exempla multiplicationis

Alia exempla

Similiter alia exempla

Habemus quidem iam tria exempla circa primum quarti capitis, in quibus ubique abstinentia, tamen quia priora manifestiora sunt, singula ista exhibuimus, nullamque fere necessariam regulam omisimus.

Exempla divisionis

Quando vero radix cubica maior, vel in radicem cubicam 4 vel radix cubica maior 5.

Similiter 81 in 3 bis ducta. 3 in 81: quorum sit quinta pars $\frac{1}{3}$, illud erit 3.

Multitudo radicis cubicae maior 24.
ista radix cubica maior 3.

Vir

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ment of cubes, cube root is to be found from the same (operations).

Examples of Multiplication

$$\sqrt[3]{11} \cdot \sqrt[3]{8} = \sqrt[3]{88} \qquad \sqrt[3]{8\tfrac{2}{3}} \cdot \sqrt[3]{\tfrac{5}{6}} = \sqrt[3]{6\tfrac{5}{9}}$$

Other Examples

$$\sqrt[3]{\tfrac{4}{9}} \cdot \sqrt[3]{\tfrac{9}{6}} = \sqrt[3]{\tfrac{1}{4}} \qquad \sqrt[3]{3\tfrac{3}{8}} \cdot 6 = 9$$

Other Similar Problems

$$\sqrt[3]{6}\ (2) = \sqrt[3]{48}, \quad \sqrt[3]{6}\ (3) = \sqrt[3]{162}, \quad \sqrt[3]{6} \cdot 4 = \sqrt[3]{384}$$

In the examples illustrating the next higher power, the fourth, where the same method is to be used, there may be some obscurity, but if the former (explanations) are remembered, it will be easy to solve these examples, and no further explanation of them will be necessary.

Examples of Division

$$\sqrt[3]{162} \div \sqrt[3]{6} = 3 \qquad \sqrt[3]{21} \div \sqrt[3]{9} = \sqrt[3]{2\tfrac{1}{3}}$$

$$\sqrt[3]{20} \div \sqrt[3]{4} = \sqrt[3]{5}$$

$$\sqrt[3]{81} \div \sqrt[3]{3} = 3 \qquad \sqrt[3]{3} \div \sqrt[3]{81} = \sqrt[3]{\tfrac{1}{3}}$$

Half of the cube root of 24 is the cube root of the number 3.

Regulares Algebræ

Sit tertia pars numeri quadrati minoris 2a.

Sunt reales radices numeri fractionis $\frac{a}{y}$

Additio et Subtractio

rx. III.

Sunt et sic considerandi duplices species, numerus surdus absolutus et irrationalis surdus absolutus. Quæ igitur communiter dicuntur inter se sunt, illæ radices non alterius additæ, vel una ab alia subtractæ, utpote in surdis quadratis tum additione tum subtractione, cum quadratum est, nisi quod alia quadrata, hæc vero omnia ingent. Quæro vero unum alterum exemplo positio, nec sunt dissimilia. Hæc vero irrationalibus et planis surdis sunt, illarum additio et subtractio per quadratissimum signum affirmativum + et negativum — absolutum.

Exemplum partis priores

Additio			Subtractio		
m/ 16	ad	m/ 54	m/ 16	de	m/ 54
8		27	8		27
2		3	2		3
5			1		
125			1		
2			2		
250			2		
m/ 250			m/ 2		

Aliud exemplum

Additio			Subtractio		
m/ $1\frac{1}{3}$	ad	m/ $4\frac{1}{2}$	m/ $1\frac{1}{3}$	de	m/ $4\frac{1}{2}$
summa m/ $10\frac{5}{12}$			manet m/ $\frac{1}{12}$		

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Also $\frac{1}{3}\sqrt[3]{24} - \sqrt[3]{\frac{24}{27}} = \sqrt[3]{\frac{8}{9}}$

Chapter III

Addition and Subtraction

Two kinds of surds are to be considered, commensurable and incommensurable. Commensurable surds are to be added or subtracted according to the above rules for the addition and subtraction of quadratic surds. (The same rules) are to be applied in cube root. Through one or two examples the method can be made clear. The addition or subtraction of incommensurable surds is more clearly understood by (annexing) the affirmative sign, +, or the negative sign, −.

An Example of the First Part

Addition [1]

$\sqrt[3]{16} + \sqrt[3]{64}$

8	27
2	3
5	
125	
2	
250	

$\sqrt[3]{250}$, the sum of roots.

Subtraction

$\sqrt[3]{54} - \sqrt[3]{16}$

27	8
3	2
1	
1	
2	
2	

$\sqrt[3]{2}$, the remainder of the roots.

Other Examples

$$\sqrt[3]{1\tfrac{1}{3}} + \sqrt[3]{4\tfrac{1}{2}} = \sqrt[3]{10\tfrac{5}{12}} \; ; \; \sqrt[3]{4\tfrac{1}{2}} - \sqrt[3]{1\tfrac{1}{3}} = \sqrt[3]{\tfrac{1}{12}}$$

[1] Recorde uses Scheubel's method as is evident in the following example:

$\sqrt[3]{81} + \sqrt[3]{24}$

3)27	8
3	2
5	
5	
125	
3	

$\sqrt[3]{375}$

Recorde, *Whetstone of witte* (1557), fol. 0 o 1, *r*.

Exempla partis posteriorum

Additio	Subtractio
m/ 18 ad m/ 25	m/ 18 de m/ 25
summa m/ 25 + m/ 18	summa m/ 25 − m/ 18

Alius exemplum

Additio	Subtractio
m/ 6 1/4 ad m/ 9 2/3	m/ 6 1/4 de m/ 9 2/3
summa m/ 9 2/3 + m/ 6 1/4	summa m/ 9 2/3 − m/ 6 1/4

Et alia additionis et subtractionis regula

Quae quis, ubi sunt numeri irrationabiles inventi. Eorum species

Sunt duo numeri irrationabiles, quorum, si primus cum primo aeris mensura vel minor quo secundus, ad ruber numerales radicandi, sicut de duo numeri radicis quam eius radicis quadratus positi. Hoc loco, utrumque radicem cum tripla quadrati radicis alterius multiplicavi, postea summa [illegible] in duobus; ab quis additio inditur, coniunge: vel, pro subtractione ab alterius numeri radicis quadrato minoris, numeri vel producti minoris radicem adde, ad illo deinde per regulas subtractionis. Dabor Quo loco, cum quod alius vel quae quam sit relinquitur, utrumque cum communi [illegible] numerabilem sunderem minoris multiplicabo [illegible] per radicis quadrati numeri ruber cum additionis, cum subtractionis etiam [illegible] erit.

Exemplum

Ubi additio	Ubi subtractio
m/ 40 ad m/ 135	m/ 40 de m/ 135
8 27	8 27
2 3	2 3
4 9	4 9
12 27	12 27
54 36	54 [illegible]

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Examples of Incommensurables

Addition

$$\sqrt[3]{18} + \sqrt[3]{28} = \sqrt[3]{28} + \sqrt[3]{18}$$

Subtraction

$$\sqrt[3]{6\tfrac{1}{4}} \text{ from } \sqrt[3]{9\tfrac{2}{3}} = \sqrt[3]{9\tfrac{2}{3}} - \sqrt[3]{6\tfrac{1}{4}}$$

Other Examples

Addition

$$\sqrt[3]{9\tfrac{2}{3}} \text{ added to } \sqrt[3]{6\tfrac{1}{4}} = \sqrt[3]{9\tfrac{2}{3}} + \sqrt[3]{6\tfrac{1}{4}},$$

Subtraction

$$\sqrt[3]{6\tfrac{1}{4}} \text{ from } \sqrt[3]{9\tfrac{2}{3}} = \sqrt[3]{9\tfrac{2}{3}} - \sqrt[3]{6\tfrac{1}{4}}$$

Another Method of Addition and Subtraction[1] of Commensurable Surds

The proposed commensurable surds are to be first reduced to rational cubes by (dividing) each surd by their common measure or number; then the roots of these cubes and the squares of these roots are to be found. Then, when each root is multiplied by three times the square of the other root, the two products are to be added to the two cubes, if addition is involved; or if subtraction is involved, when the smaller product is added to the larger cube and the larger product is added to the smaller cube, these sums are to be subtracted. When the sum or the remainder is multiplied by the common number of the commensurable surds, and the cube root of the product is taken, then the addition or the subtraction is completed.

Examples

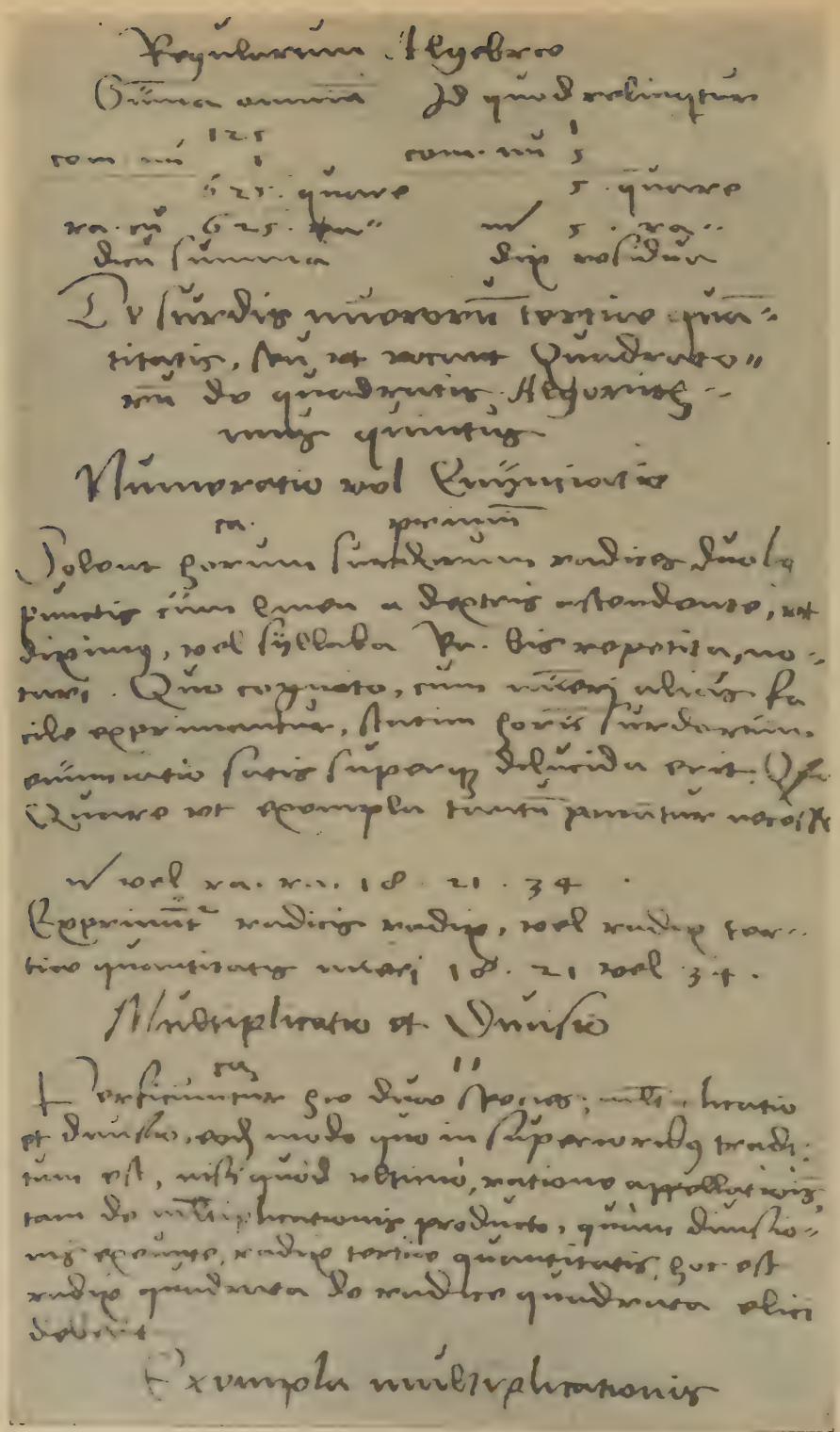
Addition

$\sqrt[3]{40} + \sqrt[3]{135}$	
8	27
2	3
4	9
12	27
54	36

Subtraction

$\sqrt[3]{135} - \sqrt[3]{40}$	
27	8
3	2
9	4
27	12
36	54

[1] Recorde, *Whetstone of witte* (1557), fol. Pp 1, *v*, prefaces his illustrations with the following: "Like differences is there in subtraction of cubike rootes commensurable. And therefore I set the examples onely, without any larger declaration."



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Sums of all	125	The remainder	1
Comm. number	5		5
	625		5

$\sqrt[3]{625}$ is the sum of the roots $\sqrt[3]{5}$ is the difference of the roots [1]

CONCERNING SURDS OF THE THIRD QUANTITY OR ROOT OF THE ROOT THE FIFTH ALGORISM

(The Fourth Root)

Chapter I

Numeration and Notation

The roots of these surds are usually indicated with the double symbolism, with lines extending from the right, as we have said, or with the syllable ra. repeated twice. With this representation in mind it is sufficiently clear from the above (explanation) how these surds should be expressed. For this reason only the necessary number of examples is given.

By the fourth root of 18, 21, and 34 is meant the root of the root (the fourth root), or the root of the third quantity of the numbers, 18, 21, and 34 ($\sqrt[4]{18}$, $\sqrt[4]{21}$, $\sqrt[4]{34}$ or $\sqrt{\sqrt{18}}$, $\sqrt{\sqrt{21}}$, $\sqrt{\sqrt{34}}$).

Chapter II

Multiplication and Division

Multiplication and division are performed as in the above (illustrations) except that here must be extracted the square root of the square root (the fourth root), or the root of the third quantity, of the product in multiplication or of the quotient in division.

[1] This problem is taken from Scheubel's 1551 edition.

Tractatus primus

w 18 m w 12 | w 18 m w 8
pro. xu. xa. 216 | pro. xu 12.

Aliud exemplum

w $1\frac{1}{3}$ m w $5\frac{4}{9}$ pro. w $3\,9\frac{1}{5}$

Alia exempla

w 8 ram 2 | w 18 ram $1\frac{1}{3}$
vel rotta pro. w 128 | pro. w $56\frac{8}{9}$

Exempla diuisionis

w 42 in w 6 | w 27 in w 3
exit w 7 | exit w 3

Alia exempla

w $4\frac{2}{9}$ in w $1\frac{1}{0}$ w $8\frac{1}{3}$ in w $5\frac{1}{3}$
ex. w $\frac{2}{3}$ ex. w $1\frac{1}{7}$

Aliud exemplum

Diuidatur w 12 in $2\frac{1}{2}$ vel rotta
exit w $\frac{1\,y\,2}{6\,2\,5}$ vel rotta $3\,\frac{4\,x}{1\,y\,2}$

Appendix, ad ea quae hactenus tum in primum etiam in praemissis algorithmi regulas de multiplicatione et diuisionibus fractionum rationalium tradita sunt, regulis nonnullis

Cum hactenus tum in hac tractatu, quemadmodo fractionibus denominatorum surdi numeri inserti, tum surdus etiam ipse rationali, vel rotta, per quae diuisi, pares tractatus docuerit, quid vero rotta irrationalis quidve eius diuisorem appellatorum surdi inserti sit ragulis praetermissis ostenderit, et alibi eius tractationem quid in praemissis de surdis descriptum sit declarare possit, prius praestabimus

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Examples of Multiplication

$\sqrt[4]{18} \cdot \sqrt[4]{12} = \sqrt[4]{216}$ $\qquad$ $\sqrt[4]{18} \cdot \sqrt[4]{8} = \sqrt{12}$

$\sqrt[4]{7\frac{1}{5}} \cdot \sqrt[4]{5\frac{4}{9}} = \sqrt[4]{39\frac{1}{5}}$

Other Examples

$2\sqrt[4]{8} = \sqrt[4]{128}$ $\qquad$ $1\frac{1}{3}\sqrt[4]{18} = \sqrt[4]{56\frac{8}{9}}$

Examples of Division

$\sqrt[4]{42} \div \sqrt[4]{6} = \sqrt[4]{7}$ $\qquad$ $\sqrt[4]{27} \div \sqrt[4]{3} = \sqrt{3}$

Other Examples

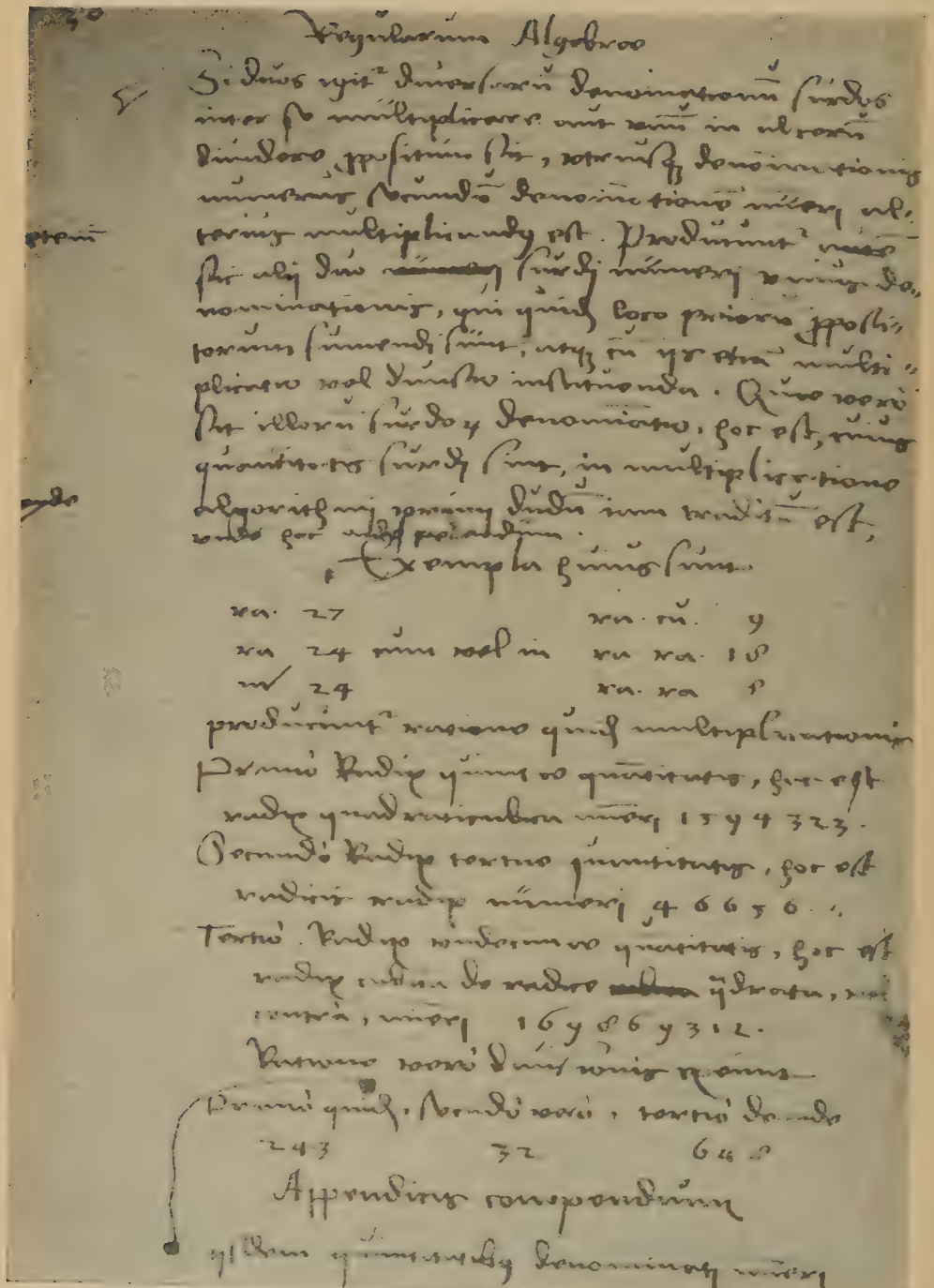
$\sqrt[4]{4\frac{7}{9}} \div \sqrt[4]{7\frac{1}{6}} = \sqrt[4]{\frac{2}{3}}$ $\qquad$ $\sqrt[4]{8\frac{1}{3}} \div \sqrt[4]{5\frac{1}{3}} = \sqrt{1\frac{1}{4}}$

Other Examples

$\sqrt[4]{12} \div 2\frac{1}{2} = \sqrt[4]{\frac{1\,9\,2}{6\,2\,5}}$ $\qquad$ $2\frac{1}{2} \div \sqrt[4]{12} = \sqrt{3\frac{4\,9}{1\,9\,2}}$

An appendix is necessary to sum up the previous explanations of multiplication and division of surds involved in this and in the preceding algorisms.

Both rational and irrational surds should be treated in two algorisms. Since surds of different denominations are usually solved according to these rules, an all inclusive treatment of the foregoing development of surds is proposed.



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If two surds of different denominations are to be either multiplied or divided, the numbers of the different denominations are changed to a common denomination through the process of multiplication. The resulting surds are multiplied or divided instead of the surds first proposed. For the denomination or quantity of these surds, refer to the rules (given) for multiplication in the first algorism.

Examples

$\sqrt{27}$	$\sqrt[3]{9}$
$\sqrt{24}$	$\sqrt[4]{18}$
$\sqrt[3]{24}$	$\sqrt[4]{8}$

The results through multiplication are:

First, the root of the fifth quantity, the quadratic cube root of 1,594,323; ($\sqrt[6]{1,594,323}$)

Second, the root of the third quantity or the root of the root of the number 46,656; ($\sqrt[4]{46,656}$)

Third, the root of the 11th quantity, the cube root of the square root of the number 169,869,312 ($\sqrt[12]{169,869,312}$)

The results through division are:[1]

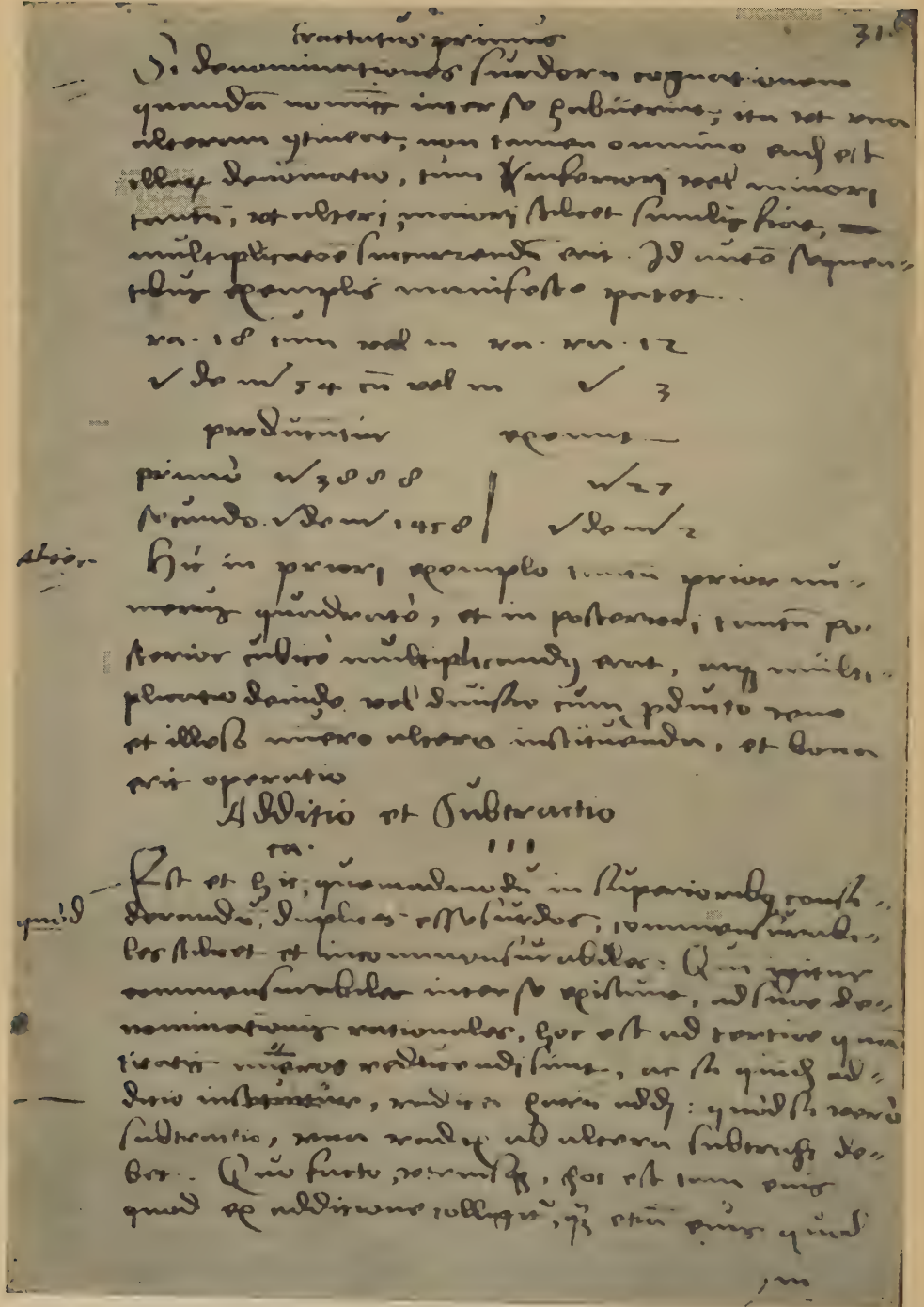
First, $\sqrt[6]{243}$

Second, $\sqrt[4]{32}$

Third, $\sqrt[12]{648}$

APPENDIX

[1] Scheubel had 243,32 and 643. Perhaps he was thinking that it was not necessary to repeat the same root signs given above.



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When surds which are not of the same denomination have a certain known relationship, such that one (power) is contained in the other, then their difference in power can be reduced through multiplication. This is illustrated by the following examples:

$$\sqrt{18} \cdot \sqrt[4]{12} = \sqrt[4]{3888} \qquad \sqrt{18} \div \sqrt[4]{12} = \sqrt[4]{27}$$

$$\sqrt[6]{54} \cdot \sqrt{3} = \sqrt[6]{1458} \qquad \sqrt[6]{54} \div \sqrt[2]{3} = \sqrt[6]{2}$$

The numbers can not be multiplied in the first example until the first number is squared, and in the second example until the second number is cubed. This gives the correct result.

Chapter III

Addition and Subtraction

Both commensurable and incommensurable roots are to be considered as in the above (examples). If (the roots) are commensurable, they should be reduced to a rational denomination, that is to numbers of the third quantity; when addition is involved, the roots of these should be added; but when subtraction is involved, one root should be taken from the other. After these operations, both the sum and the remainder, expressed as the

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third quantity, should be multiplied by the common number. Then the final result is a root of the third quantity, either the root of the sum, with the sign + attached, or the root of the remainder, completed with the negative sign.

Examples of the First Type

Addition		Subtraction	
$\sqrt[4]{6}$	+ 96	$\sqrt[4]{96}$	− $\sqrt[4]{6}$
1	16	16	1
1	2	2	1
	3		1
	81		1
	6		6
	$\overline{486}$		$\overline{6}$

$\sqrt[4]{486}$ is the sum of the roots. $\sqrt[4]{6}$ is the root of the residual.

Examples of the Second Type

Addition	Subtraction
$\sqrt[4]{8}$ added to $\sqrt[4]{18}$ = $\sqrt[4]{18}$ + $\sqrt[4]{8}$	$\sqrt[4]{8}$ from the $\sqrt[4]{18}$ = $\sqrt[4]{18}$ − $\sqrt[4]{8}$

BINOMIALS AND RESIDUALS

Algorism VI

The treatment of binomials and of residuals[1] will be simpler after their definitions have been given. (An expression) is called a binomial[2] because it consists of two names (or parts), two (so-called) rationals placed in a simple assumption, in which one

[1] "Rudolff auch den'algorithmum zu latein genennt de binomiis et residuis' indem er Euklid folgend unter Binomium ein 'zwinämig zal fürend mit ir das zeichen+. Als $5+\sqrt{7}$, unter Residuum aber 'auch ein zwinämig zal gebunden mit dem zeichen.—Als $5-\sqrt{7}$ versteht," Treutlein, *Die Coss*, p. 47.

"Das Wort Binom geht zurück auf Euklid X, 37 wo ein Ausdruck von der Form $a+\sqrt{b}$ (ex binis nominibus) genannt wird," Tropfke, *Geschichte*, I, p. 153.

[2] Treutlein says that Rudolff treats the algorism of binomials and residuals more thoroughly than his predecessors. He calls them "double numbers" and bases his development upon Euclid's *Elements*, VII.

transitus primus
manifestabilis, in Declinatum Simplex, manifestus.

6 + √11 Item 8 + √15

Appellatur nomen binomium, ex quibus duo nomina habens.

Residuum, est linea irrationalis, quum duo rationales, potentia tantum commensurabiles, possunt, quod autem non adjecta rationali, relinquunt Residuum.

6 − √11 Item 8 − √15

Nominativus vel Quinquatio

Quinquatio est recta, primum quinque lineas vel nomen primum, deinde cum eis quod adest nomen secundum, quemadmodum in superioribus tradidimus, est, quinquies

Exemplum

6 + √11 Item 8 − √21

Exprimitur 6 plus radice quadrata numeri 11. Item 8 minus radice quadrata numeri 21

Minus exemplum

√18 + √10 Item − √12.

Exprimitur radix 18 plus radix 10. et est binomium. Item radix 18 minus radix 12 et est residuum.

Additivus et Subtractivus

(Cum omnibus nunc patuit, quemadmodum unusquisque additivus et subtractivus vel subtractivus sit, utique

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(number) has a line (or a sign) of irrationality and the other (number) is joined to it (with the plus sign), such as, $6 + \sqrt{11}$, or $8 + \sqrt{15}$. The same expression indicating one number subtracted from another (number) is called a residual, $6 - \sqrt{11}$, or $8 - \sqrt{15}$.

Chapter I

Numeration or Notation

Notation is easy, for the line (of irrationality) indicates one number (or part) and the sign or symbol (+ or −) indicates the other number (or part) as illustrated in the above (examples).

Examples

$$6 + \sqrt{11} \qquad\qquad 8 - \sqrt{21}$$

The first binomial expresses 6 plus the square root of 11; the other binomial (or residual) expresses 8 minus the square root of 21.

Other Examples

$$\sqrt{18} + \sqrt{10} \qquad\qquad \sqrt{18} - \sqrt{12}$$

The first binomial expresses the square root of 18 plus the square root of 10; the second binomial or residual, expresses the square root of 18 minus the square root of 12.

Chapter II

Addition and Subtraction[1]

Since we know from the above (illustrations) how to add or subtract both absolute and surd numbers, and how to annex

[1] Scheubel (1551) makes the following geometrical application:

Primum

Esto triangulum rectangulum atque cathetus eius $8 - \sqrt{32}$, basis vero et hypothenusa simul, $16 - \sqrt{128}$, quanta erit utraque, basis scilicet hypotenusa linea seorsim qaeritur. Facit. Basis quidem $6 - \sqrt{18}$.

Hyp. vero $10 - \sqrt{50}$

Operatio

Cathetus ex hypothesi sunt $8 - \sqrt{32}$

Si autem basis 1 ra

Hyp. igitur erunt $16 - 128$ N minus 1 ra.

Regulæ Algebræ

Exemplu[m] regulæ

$6 + \sqrt{11}$ ad quod $6 + \sqrt{12}$

quorum 12 plus	quorum
radix binomij	radix residuij
$23 + \sqrt{528}$	$23 - \sqrt{528}$

Aliud exemplu[m]

$\sqrt{18} - \sqrt{10}$ ad quod $8 - \sqrt{21}$.

quorum $8 -$	quorum $8 -$
$\sqrt{21} + \sqrt{18} - \sqrt{10}$	$\sqrt{21} - \sqrt{18} + \sqrt{10}$

Exemplu[m] tertiu[m]

$8 - \sqrt{21}$ ad quod $8 + \sqrt{21}$

quorum 16. quorum $\sqrt{84}$

Exemplu[m] quartu[m]

$18 - \sqrt{1080}$ ad quod $\sqrt{1080} - 6$

quorum 12 quorum $\sqrt{720} - 24$.

Quintu[m] et ultimu[m]

$\sqrt{18} + \sqrt{12}$ ad quod $23 - \sqrt{480}$

quorum 23 plus	quorum 23 minus
radix binomij	radix residuij
$10 + \sqrt{336}$.	$660 - \sqrt{311064}$.
hoc est, plus	hoc est, minus
$\sqrt{288} + \sqrt{12}$	$\sqrt{6480} - \sqrt{12}$

Multiplicatio 111

Multiplicare singula singulis in multiplicando cum singulis quibus ipsum multiplicatur

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appropriate + and − signs, nothing more than illustrations (of binomials and of residuals) are necessary.

Examples

$$(6+\sqrt{11})+(6+\sqrt{12}) = 12+\sqrt{23+\sqrt{528}} \quad \text{or} \quad (6+\sqrt{11})-(6\sqrt{12}) = \sqrt{23-\sqrt{524}}$$

Another Example

$$(\sqrt{18}-\sqrt{10})+(8-\sqrt{21}) = 8-\sqrt{21}+\sqrt{18}-\sqrt{10} \quad \text{or} \quad \sqrt{18}-\sqrt{10} \text{ from } (8-\sqrt{21}) = 8-\sqrt{21}-\sqrt{18}+\sqrt{10}$$

A Third Example

$$(8-\sqrt{21})+(8+\sqrt{21}) = 16 \text{ or } 8-\sqrt{21} \text{ from } (8+\sqrt{21}) = \sqrt{84}$$

A Fourth Example

$$18-\sqrt{180}+(\sqrt{180}-6) = 12 \quad \text{or} \quad 18-\sqrt{180} \text{ from } (\sqrt{180}-6) = \sqrt{720}-24$$

A Fifth and Last Example

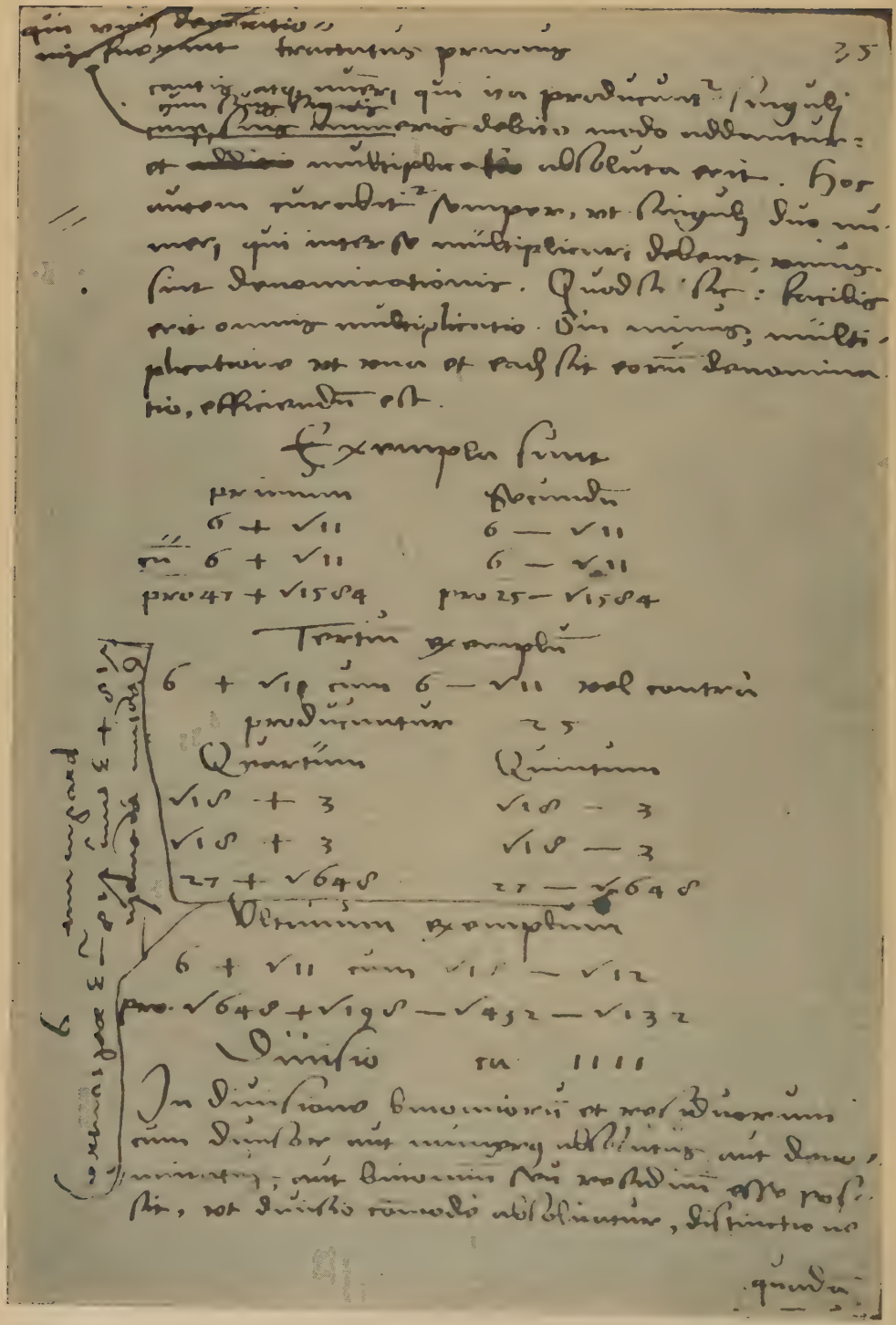
$$\sqrt{18}+\sqrt{12}+(23-\sqrt{450}) = 23+\sqrt{10+\sqrt{336}} \text{ or } 23+\sqrt{\sqrt{288}+\sqrt{12}} \quad \text{or} \quad \sqrt{18}+\sqrt{12} \text{ from } 23-\sqrt{450} = 23-\sqrt{660-\sqrt{31104}} \text{ or } 23-\sqrt{(\sqrt{648}-\sqrt{12})}$$

Chapter III

Multiplication

The different numbers of the multiplicand are multiplied[1]

[1] "Nombers that be compounde with + be called *Bimedialles* and with − *Residualles*: If Bemedialles be of two denominations they are Binomialles properly.
Since many use only Binomialles I will let the name passe. Euclides definitions doe not very aptly agree to this.
In multiplication you must multiply every part of the number by every part of the other number." Recorde, *Whetstone of witte* (1557), fol. Pp. 4, *r*.



Exempla sunt

primum	secundum
$6 + \sqrt{11}$	$6 - \sqrt{11}$
$6 + \sqrt{11}$	$6 - \sqrt{11}$
pro $47 + \sqrt{1584}$	pro $25 - \sqrt{1584}$

Tertium exemplum

$6 + \sqrt{19}$ cum $6 - \sqrt{11}$ non ducendo

producunt 25

Quantum	Quintum
$\sqrt{18} + 3$	$\sqrt{18} - 3$
$\sqrt{18} + 3$	$\sqrt{18} - 3$
$27 + \sqrt{648}$	$27 - \sqrt{648}$

Ultimum exemplum

$6 + \sqrt{11}$ cum $\sqrt{11} - \sqrt{12}$

pro $\sqrt{648} + \sqrt{198} - \sqrt{432} - \sqrt{132}$

Divisio in IIII

In divisione duorum binomiorum et non binomium cum duobus aut pluribus ablatitiis aut duorum, pluriumve binomiorum non resolutio ulla possit, aut facilis reperiri, distinctio ne

quidem

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by each number in the multiplier and the resulting numbers are added with appropriate signs and characters. The multiplication is then completed. If numbers which are to be multiplied are of the same denomination, multiplication is easy. If they are not of the same denomination, they can be reduced to the same denomination through multiplication.

Examples of Multiplication

$$(6+\sqrt{11})\ (6+\sqrt{11}) \qquad (6-11)\ (6-11)$$
$$= 47+\sqrt{1584} \qquad = 25-\sqrt{1584}$$
$$(6+\sqrt{11})\ (6-\sqrt{11}) = 25$$

$$\begin{array}{lll} \sqrt{18}+3 & \sqrt{18}-3 & \sqrt{18}+3 \\ \sqrt{18}+3 & \sqrt{18}-3 & \sqrt{18}-3 \\ \hline 27+\sqrt{648} & 27-\sqrt{648} & 9 \end{array}$$

A Last Example

$$\begin{array}{r} 6+\sqrt{11} \\ \sqrt{18}-\sqrt{12} \\ \hline \sqrt{648}+\sqrt{198}-\sqrt{432}-\sqrt{132} \end{array}$$

Chapter IV

Division

In division of binomials and of residuals, since the divisor can be an absolute number, a denominate number, a binomial, or a residual, each (division) should be worked in its own distinct way.

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As in the previous cases, the single numbers of the dividend are divided by the divisor, if it is an absolute or a denominate number, and when the single quotients are collected with appropriate + or − signs, the division is completed.

Examples of Division

$$(6+\sqrt{18}) \div 3 = 2+\sqrt{2}$$
$$(6-\sqrt{18}) \div 2 = 3-\sqrt{4\tfrac{1}{2}}$$
$$(6+\sqrt{20}) \div \sqrt{2} = \sqrt{18}+\sqrt{10}$$
$$(\sqrt{24}-2) \div \sqrt{6} = 2-\sqrt{\tfrac{2}{3}}$$

But if the divisor is a binomial or a residual, then both the divisor and the dividend should be multiplied by the name of the divisor reversed—i.e., by a residual, if (the divisor) is a binomial, or by a binomial, if (the divisor) is a residual—, and then, when these products are divided, the correct result will be obtained.[1]

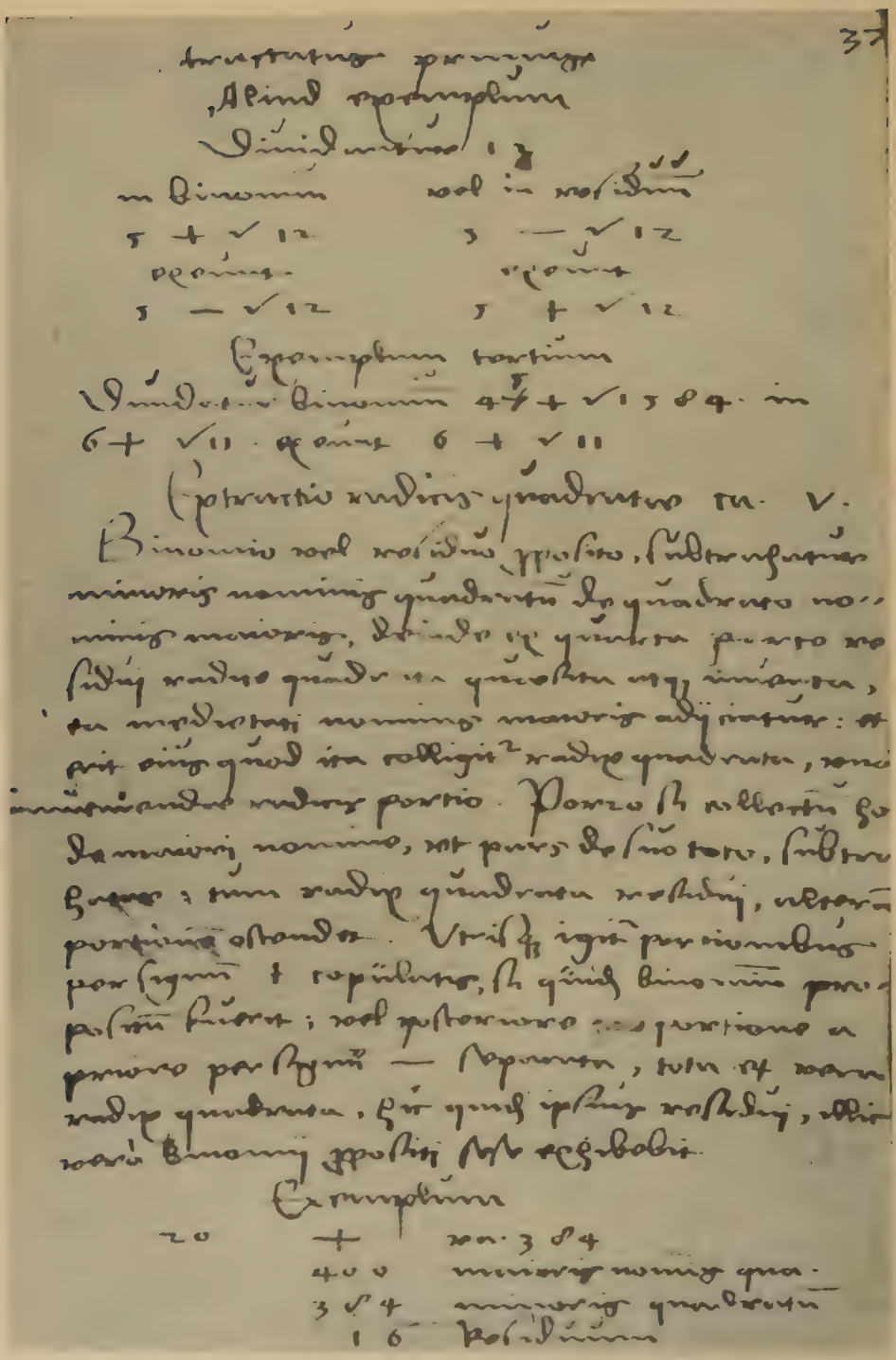
$$\frac{\sqrt{80}-\sqrt{32}}{\sqrt{8}+\sqrt{5}}$$

Multiply (both dividend and divisor) by the residual of the divisor, $\sqrt{8}-\sqrt{5}$, and the result is $\sqrt{640}-\sqrt{256}-20+\sqrt{160}$ with a dividend of 3. The divisor divided by this dividend produces

$$\sqrt{71\tfrac{1}{9}}-\sqrt{28\tfrac{4}{9}}-6\tfrac{2}{3}+\sqrt{17\tfrac{7}{9}}.$$[2]

[1] In discussing Rudolff's methods, Treutlein, *Die Coss*, p. 47, says: "Im letzten Falle soll man 'die zal so geteilt werdē sol uñ auch den teiler' mit dem Residuum bezw. Binonium multipliciren, wodruch der Theiler zu einem 'absolutischen' werde (nach Euclid, VII, 17), d. h. es sei Formel anzuwenden: $\frac{x}{\sqrt{a}\pm\sqrt{b}}=\frac{x\ (\sqrt{a}\pm\sqrt{b}.)''}{a-b}$

[2] Scheubel had $\sqrt{5120}-\sqrt{2048}-20+\sqrt{160}$ or $\sqrt{568\ 8/9}-\sqrt{227\ 5/9}-6\ 2/3+\sqrt{17\ 7/9}$.



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Another Example

Divide 13 by the binomial $5 + \sqrt{12}$ or by the residual $5 - \sqrt{12}$. The quotient is $5 - \sqrt{12}$ or $5 + \sqrt{12}$.

A Third Example

Divide the binomial $47 + \sqrt{1584}$ by $6 + \sqrt{11}$. The quotient is $6 + \sqrt{11}$.

Chapter V

Extraction of Square Root[1]

When a binomial or a residual is given, subtract the square of the lesser number from the square of the larger (number), divide the remainder by 4, and add to the square root of this result half of the larger number. One part of the root will be the square root of this sum. Subtract the sum just found from the larger number and the square root of the remainder will be the other part (of the root). Separate both parts (of the root) with the sign plus, if it is a binomial, or by the sign minus, if it is a residual. The square root is then completed.[2]

Examples of Extraction of Square Root

$20 + \sqrt{384}$

400, the square of the larger number.
384, the square of the smaller number.
16 remaining.

[1] "Cum haec ex 17 propositione Euclidis lib. septimi eandem quam ipsi multiplicati, hoc est, dividendus et divisor propositi rationem custodiant," is added in Scheubel's *Algebra*, 1551.

[2] Scheubel's 1551 edition says: "Et haec quidem de Binomiorum radicibus inveniendis dicta sufficiant. Simili modo iam agendum est cum Residuis, cum et ipsa quadrata esse, atque ita, radices quadratus habere, ex propositione 91 et ordine sequentibus quinque eius dem decimi Euclidis manifeste poteat. Quare pro iis eadem modo operatione instituta."

Regulæ Algebræ

4 quarta pars roth 2
2 radix ad 10
summa 12 cubica radix
quadrata √12 portio una
Quæro √8 portio altera.

Atq; sumi √12 + √8 proportio binomij radicis quadratæ.

Aliud exemplum

27 — æq. 64 8

729 maioris numeri quadratum
648 minoris quadratum
81 Rothomagum
20 1/4 rothomagi quarta pars
4 1/2 radix ad 13 1/2
summa 18 cubica radix
quadrata √18 portio una
Quæro 3 portio altera

Atq; sumi √18 — 3 proportio rothomagi radicis quadratæ.

Tertium exemplum

æq. 10 — æq. 10

10
10
8
2
√2 ad √4 1/2
summa √12 1/2 cubica radix
√12 1/2 portio una
Quæro √1/2 portio altera

Atq; sumi √12 1/2 — √1/2 proportio rothomagi radicis quadratæ. Id quod re ami-
mi patet.

Quarta

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$16 \times \frac{1}{4} = 4$ The root 2, added to 10 = 12

$\sqrt{12}$, one part of the root.

$\sqrt{8}$, the other part of the root.

$\sqrt{12} + \sqrt{8}$, the square root of the proposed binomial.

Another Illustration

$$27 - \sqrt{648}$$

729, the square of the larger number.

648, the square of the smaller number.

81, the remainder.

$81 \times \frac{1}{4} = 20\frac{1}{4}$; $\sqrt{20\frac{1}{4}} = 4\frac{1}{2}$; $4\frac{1}{2} + 13\frac{1}{2} = 18$

$\sqrt{18}$, one part of the root.

$27 - 18 = 9$; $\sqrt{9} = 3$, the other part of the root.

$\sqrt{18} - 3$, the square root of the proposed residual.

A Third Illustration

$$\sqrt{18} - \sqrt{10}$$

18, the square of the larger number.

10, the square of the smaller number.

8, the remainder.

2

$\sqrt{2} + \sqrt{4\frac{1}{2}} = \sqrt{12\frac{1}{2}}$; $\sqrt{\sqrt{12\frac{1}{2}}} = \sqrt[4]{12\frac{1}{2}}$, one part.

$\sqrt[4]{\frac{1}{2}}$, the other part.

$\sqrt[4]{12\frac{1}{2}} - \sqrt[4]{\frac{1}{2}}$, the square root of the residual.

This can be proved.

tractatus primus

Quintum exemplum

xx. 1 ℘ + 3

1 ℘ quadrata maioris, y minoris
2 1/4 rotidij quinta pars. 1 1/2 ipa radicę.
Radix binomij 1 1/2 + √4 1/2 partis maioris
Radix residij 1 1/2 − √4 1/2 partis minoris
Atq; tandem proposita binomij
tota et vera radix
Radix reminoris, cum residui
binomij 1 1/2 + √4 1/2 quum reb. 1 1/2 − √4 1/2

Regulares algebrae tractatus
secundus, qui est de aequationibus
et primum de amplis

Comprehendimus in hoc secundo tractatu
aequationes tres, tumq; potiores et magis pro-
prias, quas inter ordine in hoc exemplo
describamus

Aequatio prima

Prima aequatio est, quando binae species
eiusdem cum suis numeris aequales,
et characterum eiusdem aliquis aequat. In hac aequa-
tione, pro radicis valore consequendo, nume-
rus characteris inferioris ad habitudinem nu-
merum characteris superioris ad significa-
tionis dividit, et totius numerus quem
significat, characteris valorem reperimus
Quia si radix est, radicis valor ac primitus
[illegible] et aequatur

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A Fourth Example

$\sqrt{18} + 3$

18, the square of the larger number; 9, the square of the smaller number.

$2\frac{1}{4}$, a fourth part of the difference of the squares and $1\frac{1}{2}$, the square root of $2\frac{1}{4}$.

The root of the binomial $1\frac{1}{2} + \sqrt{4\frac{1}{2}}$, one part, $\sqrt{1\frac{1}{2} + \sqrt{4\frac{1}{2}}}$

The root of the residual $1\frac{1}{2} - \sqrt{4\frac{1}{2}}$, the other part, $\sqrt{1\frac{1}{2} - \sqrt{4\frac{1}{2}}}$

Therefore the whole or real root of the proposed binomial consists of both the root of the binomial $1\frac{1}{2} + \sqrt{4\frac{1}{2}}$, and the root of the residual $1\frac{1}{2} - \sqrt{4\frac{1}{2}}$.

A SECOND TREATISE OF THE RULES OF ALGEBRA CONTAINING EQUATIONS AND A FEW ILLUSTRATIONS

In the second treatise three equations are to be explained, and illustrated in the order in which they occur.

The First Equation

In the first equation,[1] composed of two characters, the sign, value and number of one character satisfy, or equal (the sign, value and number of) the other character. To find the value of the accompanying root in this equation, the number[2] of the smaller character, or of the character with the lesser power, is divided into the number of the higher character, or of the character with the greater power, and the resulting quotient expresses this value or root. If there is only one root (to be extracted), when the value of the root is expressed, the equation is satisfied.[3]

[1] Scheubel's *Algebra* of 1551 gives the following definition, p. 12, *v*: "*Est Aeque Prima Aequatio*, in quae unius quantitatis vel characteris numerus unius characteris numero aequatur. Secundo vero et tertia aequationes sunt, ubi tribus characteribus consignatis numeris, illic quidem naturali eorum ordine, hic vero iam una iam duobus vel pluribus, observato ordine tenterrupto, omnis characteribus numeri duorum uni, vel contra unius characteris numerus duobus aequatur. Et de his tribus nunc deinceps ordine dicemus et primo quidem de processu aequationis primae.

Prima aequation est, ubi duae quantitates vel duo numeri, diversis characteribus signati, inter se aequales esse proferuntur."

Recorde, *Whetstone of witte* (1557), fol. F f 1, *r*, compares his treatment of equations with Scheubel's: "This rule of equation, is divided by some men, into diverse partes. As namely Scneubelius dooeth make 3 rules of it. And in the second rule he putteth 3 severalle cannons. Some other men make a greater nomber of definitions in this rule. But I intende (as I thinke beste for this treatice, which may serve as farre as their workes do extende) to divide it onely into twoo partes. Whereof the first is, when one nomber is equalle unto one other. And the seconde is, when one nomber is compared as equalle unto 2 other nombers."

[2] The coefficient was called by Diophantus (c. 275) the *plêthos* (multitude) and by Brahmagupta (c. 628) the *anca* or *pracriti*. The word "Coefficient" originated with Vieta; Colebrooke, *Algebra*, pp. 246, 348.

[3] It is interesting to compare this development with Recorde's "Rule of equation commonly called Algebers Rule: When any question is propounded, apperteinyng to this rule, you shall imagine a name for the nomber, that is to bee soughte, as you remember, that you learned in the rule of false position. And with that nomber shall you procede accordyng to the question until you finde a Cossike nomber that the question expresseth," *Whetstone of witte* (1557), fol. E e 4, *v*.

Regula Algebræ
seu Rhetorum est. Quod si vero alia quæ-
ratur, tum eq̃ numero eq̃ito, per eadem
valores comparando, radix secundæ quæsi-
tionis eq̃ationis elicienda est, et eq̃uationi
similiter satisfactum erit.

Eq̃uationes exemplo

3 radices } 16 N.
3 primis } eq̃antur 6 ra.
3 tertiis } 6 ter.

Finita diuisione, reperitur in singulis 2 N.
valorem radicis indicare. Quod eq̃-
uisse probat

Alia exemplo

3 primos eq̃antur 12 N.
3 secundos 12 ra.

in eq̃o vero finita diuisione, reperitur 4, primos quæ-
rentis valor, ut tunc ipsis radicis. Quod eq̃uisse probat

Similiter alia exemplo

$2\frac{1}{3}$ secundos eq̃antur 63 N.
$2\frac{1}{3}$ ter. $37\frac{1}{2}$ N.
$\frac{4}{5}$ quar. $31\frac{1}{4}$ ra.
$1\frac{1}{9}$ quin. $\frac{125}{187}$ ra.

Finita diuisione, erit eq̃uationis radicis valor
in exemplo quinti primo, eq̃uatio radicis
alicuius vocat radix secundo quæsitorum: in
secundo vero et tertio, eq̃uatio radicis vocat
dicit, per est radix tertiæ quæsitorum: ut
tunc in exemplo quarto, eq̃uatio radicis
quartæ quæsitorum radicis valor erit

Finis.

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But if instead of the resulting value of the root another quantity can be extracted, then this value can be tested as (the real value of) the root and similarly the equation is satisfied.

Examples

$3x = 6$ $\qquad x = 3$
$3x^2 = 6x$ $\qquad x = 2$
$3x^4 = 6x^3$ $\qquad x = 2$

In each case the result of division, or the value of the root, is 2. This can be proved.

Other Examples

$3x^2 = 12$ $\qquad 3x^3 = 12x$

The result of division is 4, a value of the first quantity, and 2 is the value of this root. This can be proved.

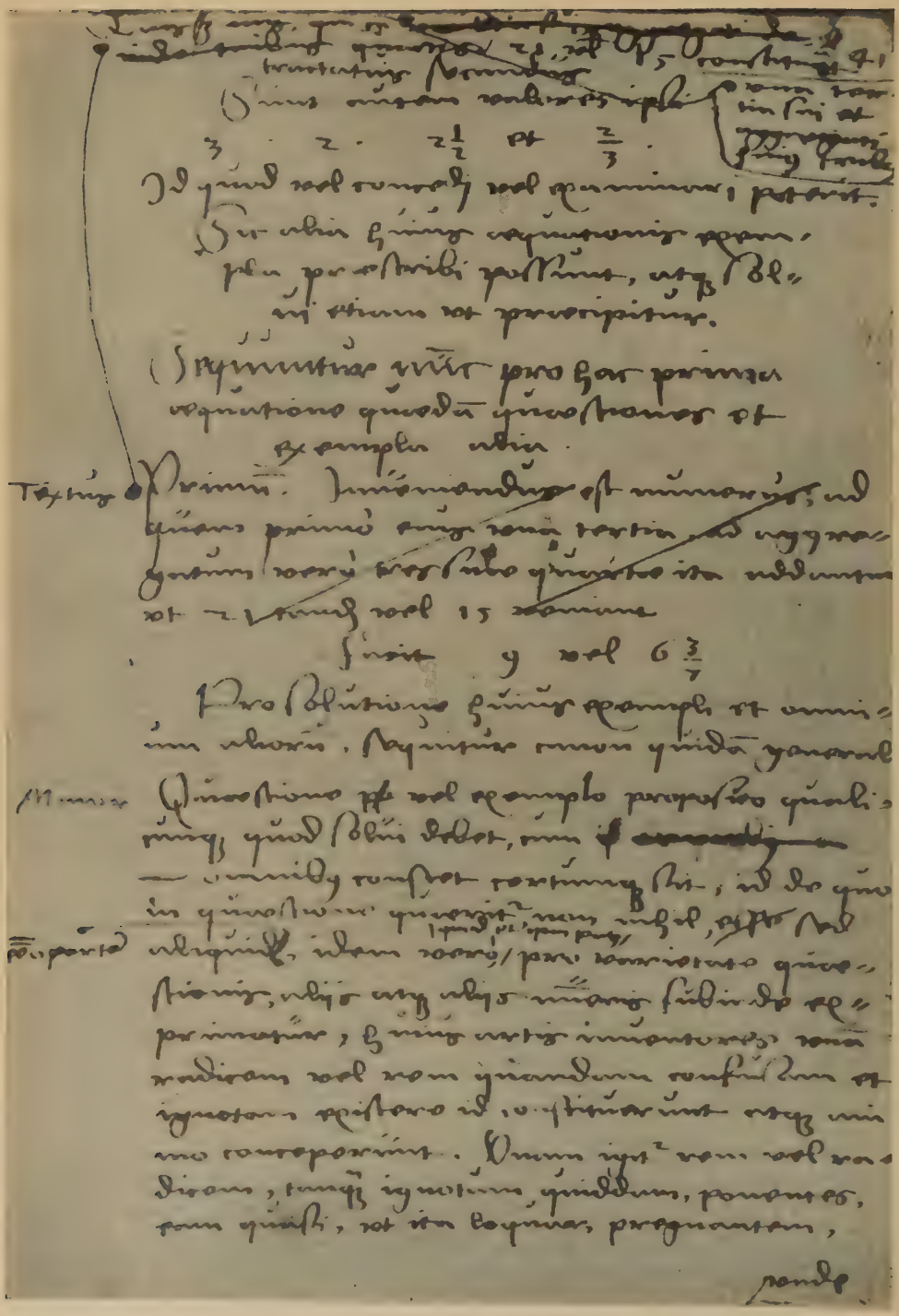
Other Similar Examples

$2\frac{1}{3}\, x^3 = 63$
$2\frac{1}{3}\, x^4 = 37\frac{1}{3}$

$\frac{4}{5}\, x^5 = 31\frac{1}{2}x$

$7\frac{1}{9}\, x^6 = \frac{2048x}{2187}$

After dividing, the value of the root in the first example is a cube root, or a root of the second quantity; in the second and third (examples) the value is a root of a root, or a root of the third quantity; and in the fourth example the value of the root is a sursolid, or a root of the fourth quantity.



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The resulting values, 3, 2, $2\frac{1}{2}$ and $\frac{2}{3}$, can be tested or proved.

Similarly other illustrations of the (first) equation can be given and proved.

Certain Questions and Other Illustrations of the First Equation

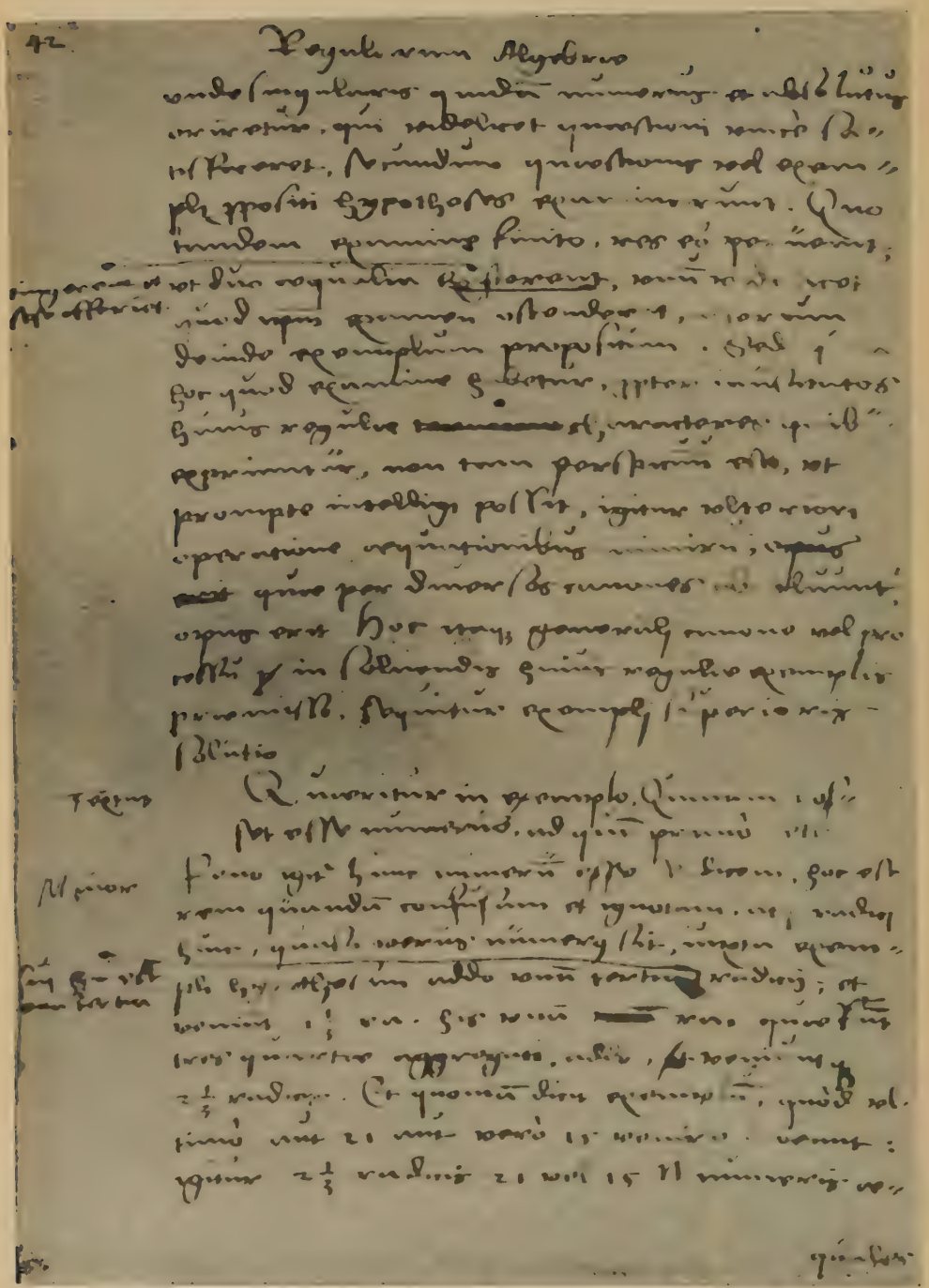
Find a number such that the sum of that number and its third plus three fourths of this sum will equal 21 or 15.

$$\tfrac{7}{3}\,x = 21 \text{ or } x + \tfrac{4}{3}\,x = 21;\ x = 9$$
$$\tfrac{7}{3}\,x = 15 \text{ or } x + \tfrac{4}{3}\,x = 15;\ x = 6\tfrac{3}{7}$$

A general rule follows for the solution of this example and of all other (examples of this type):

When it is clear from certain definite statements that some particular value is desired as a solution for a given question, or for a certain type of example, test one number after another in a series. When the inventors[1] of this art conceived of a single thing (or value), involved as a root or as an unknown in an equation, they tested certain absolute numbers to see if they completely satisfied the equation

[1] Recorde, *Whetstone of witte* (1557), fol. E e 4, *v*, "The rule of equation is called the Rule of Algebra after the name of the inventoure as some men thinke: or by a name of singular excellencie, as others judge. But of this use it is rightly called, the rule of equation: bicause that by equation of nombers, it doeth dissolve doubtfull questions: and unfolde intricate ridles."



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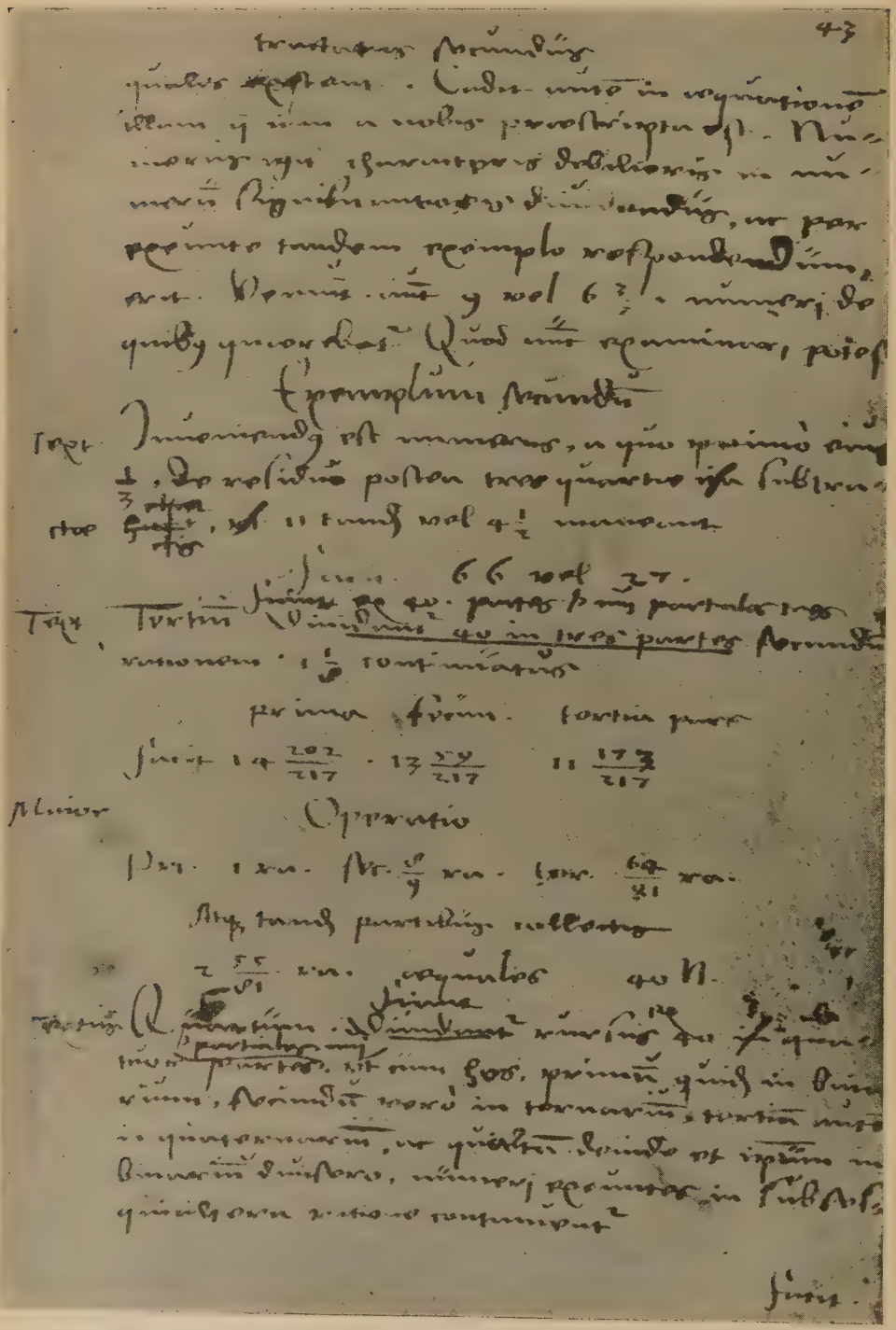
formed from the given hypothesis. They found as a result that two equals exist, one resulting from the test, the other from the given example. But because this rule is not sufficient to make clear (the value) resulting from the test, the solution of an equation, according to a general rule, is necessary to make it easily understood. Thus the solution[1] of the above example is given as a preface to the general solution of problems of this type.

It is asked in the example, "What are the numbers which first etc. . . . ?"

I place this number to be the root, the certain, unknown thing, and to this root, as if it were the real number (or value) according to the hypothesis of the example, I add one third of it, i.e. $\frac{1}{3}$ of the root, and I get $1\frac{1}{3}$ root ($1\frac{1}{3}x$); to this I add one root, or $\frac{3}{4}$ of my former sum, and then I have $2\frac{1}{3}$ times the root ($\frac{7}{3}x$). Since the example says that the value is 21 or 15, then the result is 21 or 15.

[1] In his 1551 edition, Scheubel says: "Aenigmata seu quaestiones quorum solutiones, tandem hanc primum aequationem requiret," fol. 13, *r*. At the end of his solutions he remarks: "In omnibus exemplis quae per has Algebre regulas solvi debet, loco eius qui quaestioni satisfacere debeat numeri, Radix seu una res, tanquam aliquid id esse de quo quaeritur significans, ponenda est: ea deinde ac si versus solutions numerus esset," fol. 13, *v*.



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This is the correct value in the given equation. When the number of the smaller character is divided into the number of the larger character, the value of the quotient will satisfy (the equation) in the given example. Thus, 9 and $6\frac{3}{7}$ are the required numbers. This can be proved.

Second

Find a number such that the difference between the remainder when $\frac{1}{3}$ is taken away and $\frac{3}{4}$ of this remainder is 11 or $\frac{9}{2}$. The result is 66 or 27.

Third

Divide 40 into 3 parts[1] which are in the successive ratio of $1\frac{1}{8}$.

First Part	Second Part	Third Part
$14\ \frac{202}{217}$	$13\ \frac{59}{217}$	$11\ \frac{173}{217}$

Operation

The parts, x, $\frac{8}{9}x$, $\frac{64}{81}x$

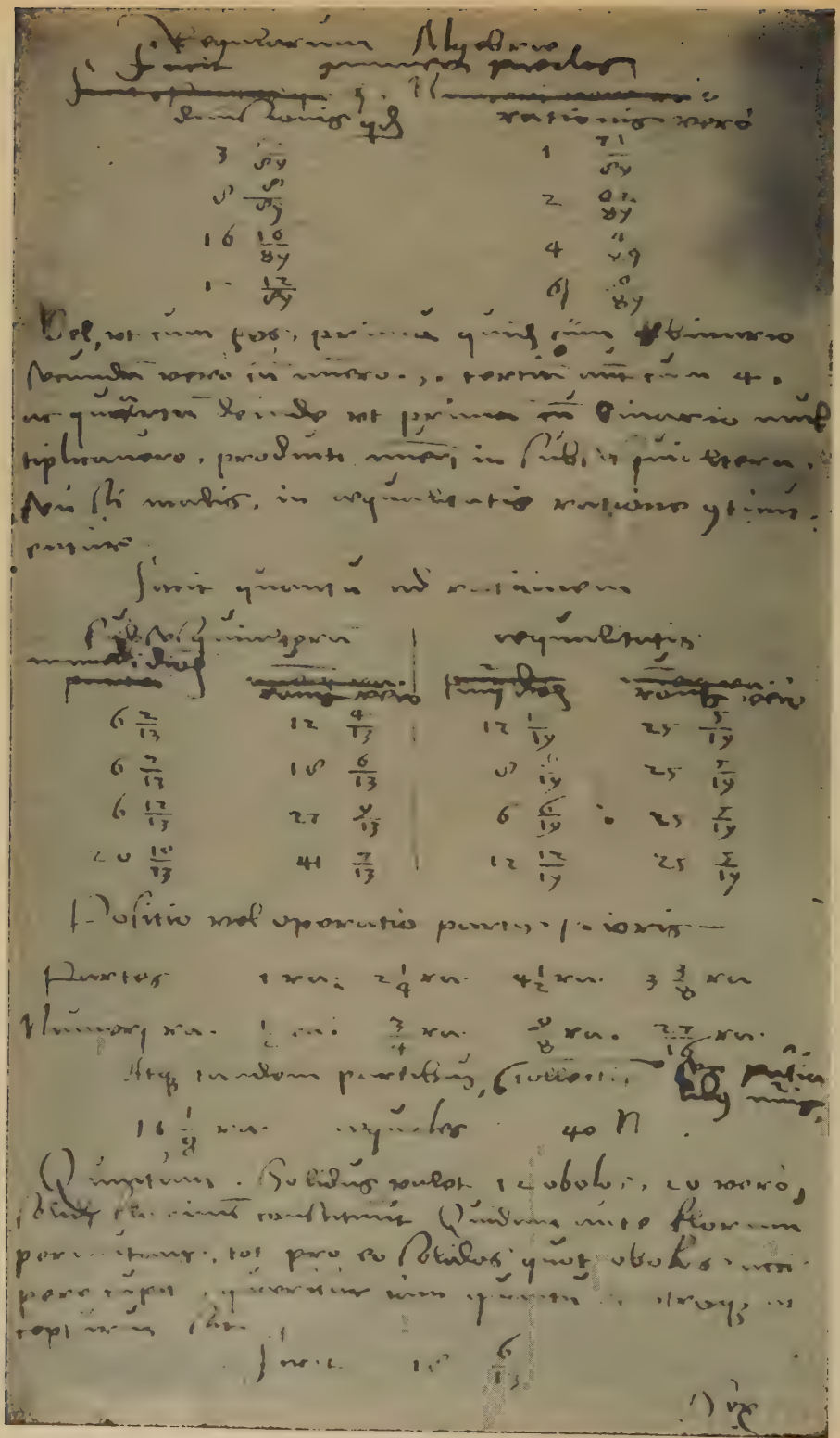
When the parts are collected the result is $2\frac{55}{81}x = 40$.

Fourth

Divide 40 into 4 parts such that when the first (part) is divided by 2, the second (part is divided) by 3, the third (part is divided) by 4, and the fourth (part is divided) by 2, the resulting numbers are in a continuous ratio of one and one half.

[1] Scheubel, in his 1551 edition, gives two problems concerning the division of the number 40. In fol. 14, *r*, he says: "Dividantur 40 in tres partes secundum rationem subsuperbipartientem tertias continuatas.

$7\frac{17}{49}$, $12\frac{12}{49}$, $20\frac{20}{49}$."



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The Divisions	The Ratios
$3\frac{5}{8}\frac{3}{9}$	$1\frac{7}{8}\frac{1}{9}$
$8\frac{8}{8}\frac{}{9}$	$2\frac{6}{8}\frac{2}{9}$
$16\frac{1}{8}\frac{6}{9}$	$4\frac{4}{8}\frac{}{9}$
$12\frac{1}{8}\frac{2}{9}$	$6\frac{6}{8}\frac{}{9}$

Now when these are multiplied in the following manner, the first by 2, the second by 3, the third by 4 and the fourth by 2, then the numbers produced are in a successive ratio of one and one half, or if you prefer, they are in a continuous equal ratio.

Numbers in the ratio of one and one half. The equal ratio.

$6\frac{2}{1}\frac{}{3}$	$12\frac{4}{1}\frac{}{3}$	$12\frac{1}{1}\frac{2}{9}$	$25\frac{5}{1}\frac{}{9}$
$6\frac{2}{1}\frac{}{3}$	$18\frac{6}{1}\frac{}{3}$	$8\frac{8}{1}\frac{}{9}$	$25\frac{5}{1}\frac{}{9}$
$6\frac{1}{1}\frac{2}{3}$	$27\frac{9}{1}\frac{}{3}$	$6\frac{6}{1}\frac{}{9}$	$25\frac{5}{1}\frac{}{9}$
$20\frac{1}{1}\frac{0}{3}$	$41\frac{7}{1}\frac{}{3}$	$12\frac{1}{1}\frac{2}{9}$	$25\frac{5}{1}\frac{}{9}$

The condition or operation of the first part.

Parts, x, $2\frac{1}{2}x$, $4\frac{1}{2}x$, $3\frac{3}{8}\ x$.

Root numbers, $\frac{1}{2}x$, $\frac{3}{4}x$, $\frac{9}{8}x$, $\frac{2}{1}\frac{7}{6}x$

When the parts are added, we have $11\frac{1}{8}x = 40$; $x = 3\frac{5}{8}\frac{3}{9}$

Fifth

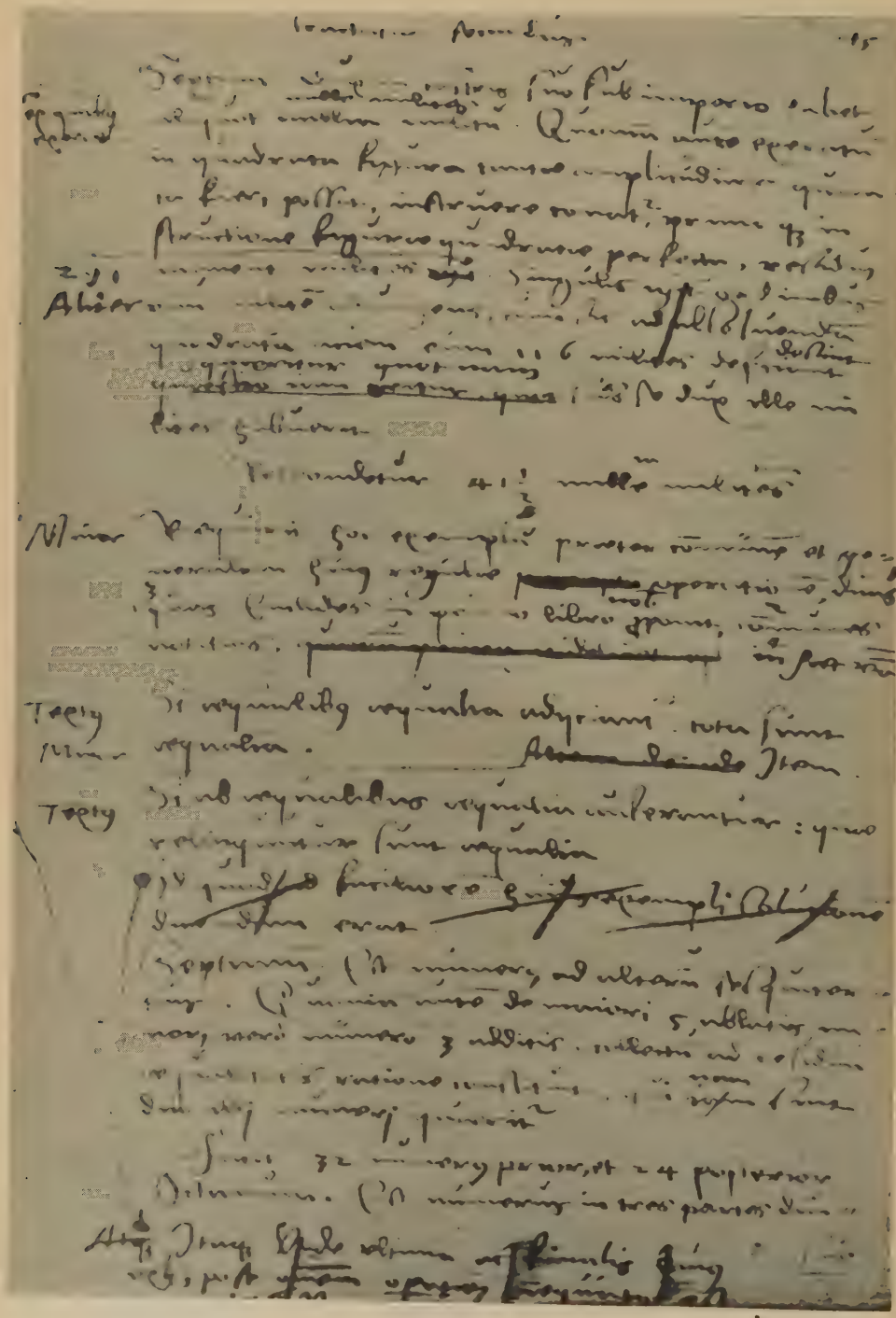
One solidus contains 12 oboli, one florin contains 20 solidi. Someone desires to change 1 florin into equal parts of solidi and oboli. How many parts of each does he obtain? Answer $18\frac{6}{1}\frac{}{3}$.

[1] The fourth problem has the following solution:

Parts	Ratios
$x = 3^{5}\frac{3}{8}\frac{}{9}$	$x\frac{1}{2} = 1^{7}\frac{1}{8}\frac{}{9}$
$\frac{3}{4}x = 8\frac{8}{8}\frac{}{9}$	$\frac{3}{4}x = 2^{6}\frac{2}{8}\frac{}{9}$
$3^{6}x\frac{9}{8} = 16^{1}\frac{6}{8}\frac{}{9}$	$\frac{9}{8}x = 4\frac{4}{8}\frac{}{9}$
$2^{7}\frac{9}{8}x = 12^{1}\frac{2}{8}\frac{}{9}$	$2^{7}\frac{9}{1}\frac{}{6}x = 6\frac{6}{8}\frac{}{9}$
$^{8}\frac{9}{8}x = 40$	
$x = 3^{5}\frac{3}{8}\frac{}{9}$	

Numbers	Ratios
$2x = 6\frac{3}{1}\frac{}{3} = (x')$	$4x = 12\frac{6}{1}\frac{}{3}$
$2x = 6\frac{3}{1}\frac{}{3} = (x'_2)$	$6x = 18\frac{9}{1}\frac{}{3}$
$^{9}x\frac{}{4} = 6^{1}\frac{2}{1}\frac{}{3} = (^{9}x'_{4})$	$9x = 27\frac{9}{1}\frac{}{3}$
$^{27}x\frac{}{4} = 41\frac{7}{1}\frac{}{3} = (^{27}\frac{}{8}x')$	$^{27}x\frac{1}{2} = 41\frac{7}{1}\frac{}{3}$
$^{52}\frac{}{4}x$ or $13x = 40$ $(^{52}x\frac{1}{8} = 40)$	
$x = 3\frac{1}{1}\frac{}{3}$ $(x' = 6\frac{2}{3})$	

Numbers	Ratios
$x = 12\frac{1}{1}\frac{2}{9}$	$2x = 25\frac{5}{1}\frac{}{9}$
$\frac{2}{8}x = 8\frac{8}{1}\frac{}{9}$	$2x = 25\frac{5}{1}\frac{}{9}$
$x\frac{1}{2} = 6\frac{6}{1}\frac{}{9}$	$2x = 25\frac{5}{1}\frac{}{9}$
$x = 12\frac{1}{1}\frac{2}{9}$	$2x = 25\frac{5}{1}\frac{}{9}$
$^{1}\frac{9}{8}x = 40$	
$x = 12\frac{1}{1}\frac{2}{9}$	



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Sixth

A general has several thousand soldiers under his command. When he attempts to arrange his army in the greatest possible square,[1] he finds that the construction of this square leaves out 291 soldiers. If he then adds a single soldier to each battle line in this square he lacks 116 soldiers. How many soldiers did the general have in his command?

(In this example, instead of saying single I should say add one soldier to each line—of what the historian calls a phalanx—and then he will lack 116 soldiers.)

After (performing) all of the operations (indicated) in this example, the number of soldiers in each line, or the root, is represented by $2x = 406$. The result is $41\frac{1}{2}$ thousands of soldiers.

Besides the common and general analogy of the rule (for the first equation), this example requires two famous axioms which Euclid proposes in his first book. (These are):

1. If equals are added to equals the sums are equal.
2. If equals are subtracted from equals the remainders are equal.

Seventh

A number is one and one third times another. If five is subtracted from the larger (number) and if three is added to the smaller (number), the remainder and sum are equal. What are the two numbers? The first number is 32, and the second number is 24.

Eighth

A number is divided into three parts.

[1] Recorde not only quotes Scheubel, but he also uses about the same type of problems. The following is an interesting illustration: "There is a capitain, which hath a greate armie, and would gladly marshall them, into a square battaile, as large as aughte bee. Wherefore in his firste proofe of square forme, he had remainyng 284 to many. And prouyng again by putting 1 moare in fronte, he founde wante of 25 men. How many souldiars had he, as you gesse? x=154," *Whetstone of witte* (1557), fol. G g 3, *r*.

Regula Algebrae

Primus

Tertij

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Quoniam autem prima pars et 11 respectu numeri divisi, rationem aeqũalitatis: secunda vero, subscriptam: ac tertia deinde, et ipsa respectu divisi (si tamen prius $3\frac{1}{2}$ aliunde acciperit vel tantum amiserit) subsesquitertiam rationem instituit, quantus iam sic totus numerus, quantae etiam singulae partea quaeritur.[1]

Facit respectu eius quia tertia accedunt $13_{1\,3}^{5}$, $2_{1\,3}^{5}$, $4_{1\,3}^{6}$, $6_{1\,3}^{7}$
tertius amictit $6_{1\,3}^{1\,2}$, $4_{1\,3}^{1}$, $2_{1\,3}^{4}$, $8_{1\,3}^{9}$

Operation

One root or the whole number, (is x)
$x - 11$, the first part.
$\frac{1}{2}x$, the second (part)
$\frac{3}{4}x - 3\frac{1}{2}$, or $\frac{3}{4}x + 7\frac{1}{2}$, the third part.
Then $2_{1\,2}^{1}x - 14\frac{1}{2}$, or $2_{1\,2}^{1}x - 7\frac{1}{2}$, equals one root ($= x$)

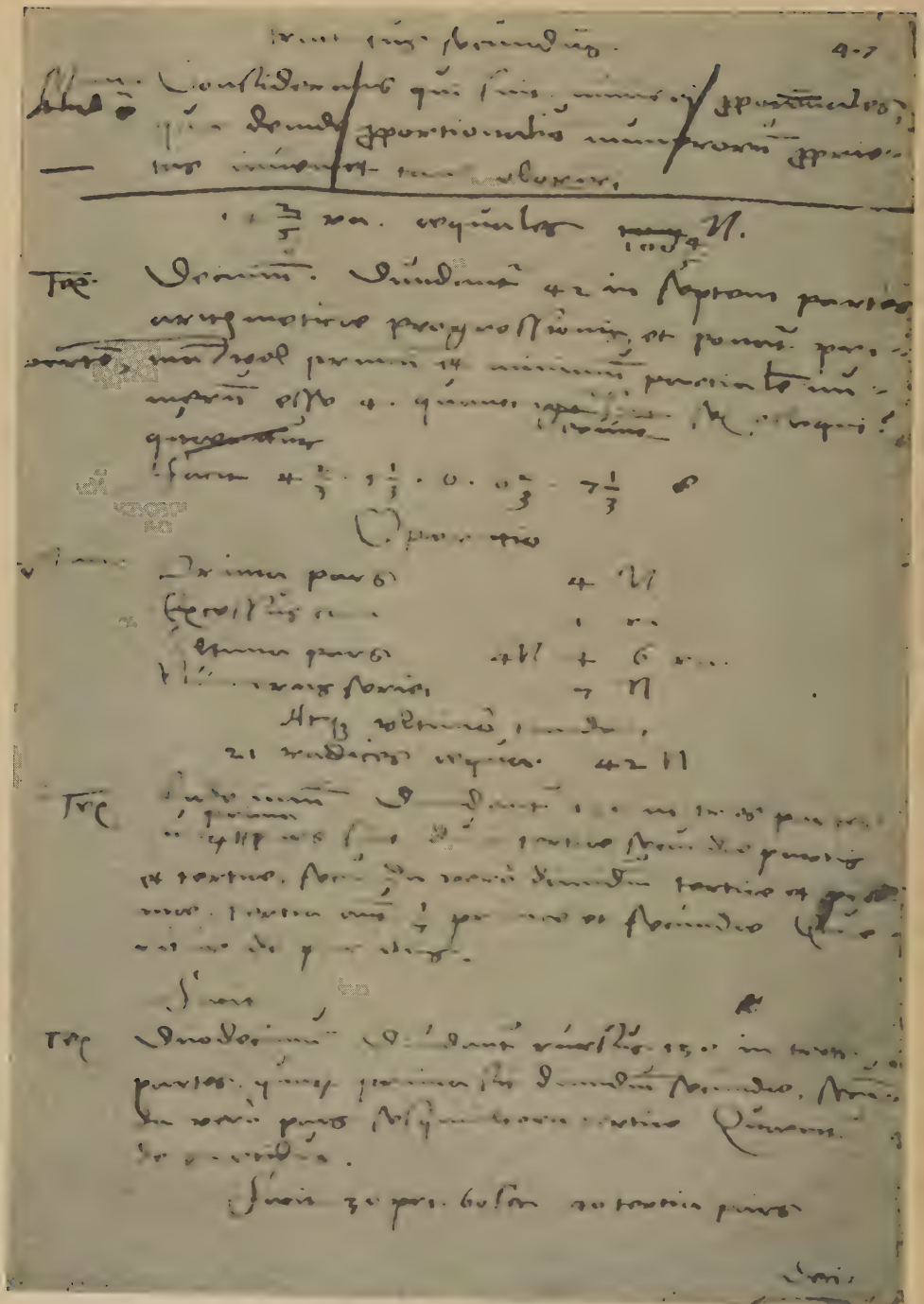
Adding and subtracting, the result is:

$1_{1\,2}^{1}x = 14\frac{1}{2}$ $\qquad$ $(x = 13_{1\,3}^{5})$
$1_{1\,2}^{1}x = 7\frac{1}{2}$ $\qquad$ $(x = 6_{1\,3}^{1\,2})$

Ninth

The first of three numbers is $8\frac{2}{5}$, and the third is 6; the first and third have the same ratio that the excess of the first over the second has to the excess of the second over the third. Find the second number which is unknown. The number is 7.

[1] Since no translation of this statement agrees with Scheubel's explanation, probably he intended to substitute another problem before the work was printed.



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$$14\tfrac{2}{5}x = 100\tfrac{1}{5}$$
$$x = 7$$

This can be proved.

Tenth

Divide 42 into 7 parts which are in an arithmetical progression. If the first and smallest part is 4, what are the six remaining numbers?

$4\frac{2}{3}$, $5\frac{1}{3}$, 6, $6\frac{2}{3}$, $7\frac{1}{3}$, 8.
First part 4.
Common excess $1x$ (or the difference, $\frac{2}{3}$).
Last part $4 + 6x$.
Number of terms in the series, 7.
$21x + 28$[1] $= 42$.

Eleventh

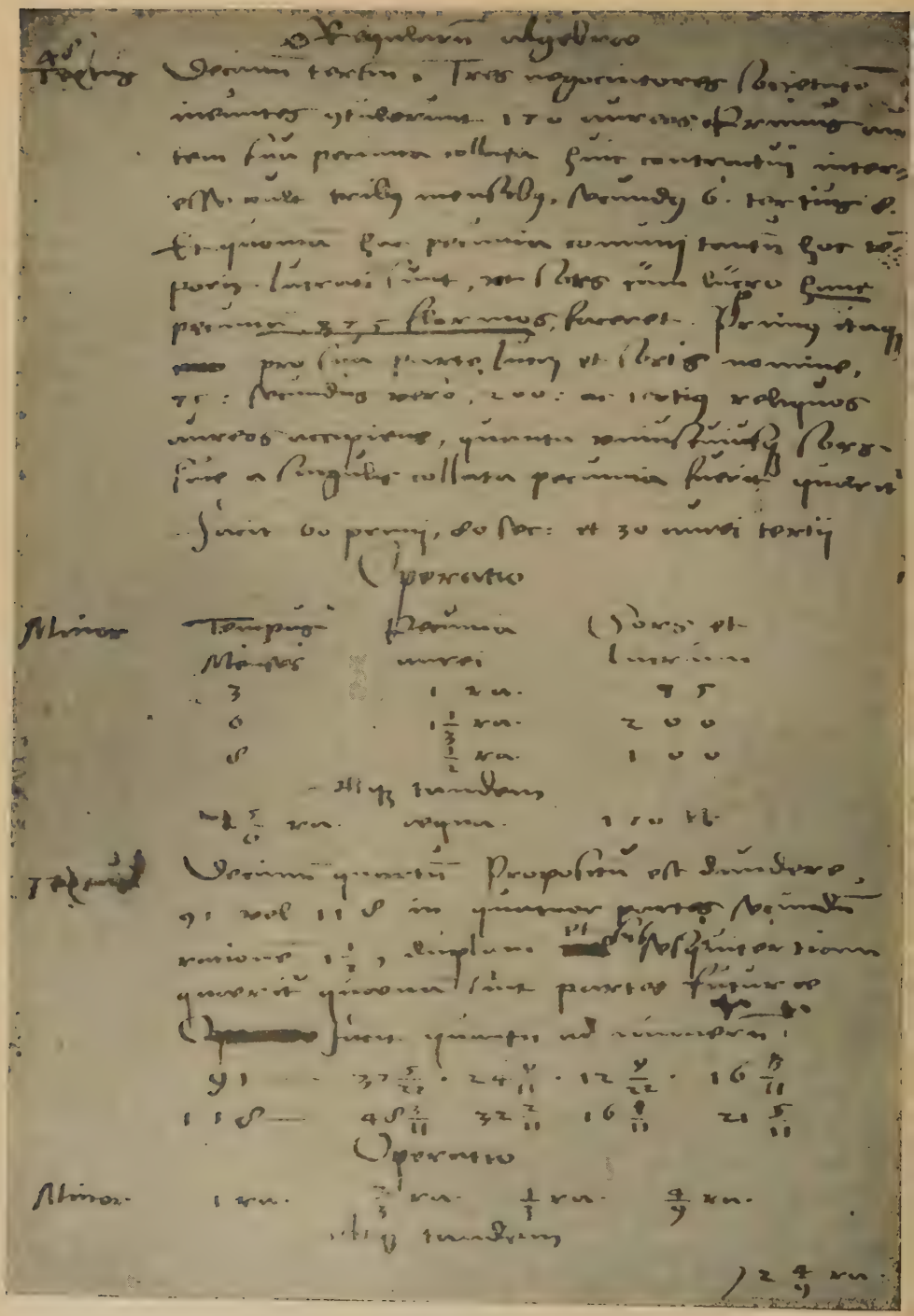
Divide 130 into three parts such that the first (part) is $\frac{2}{3}$ of the sum of the second and third (parts), the second (part) is one half of the sum of the third and first (parts), and the third (part) is one seventh of the sum of the first and second (parts). Find the parts. (52, 18, 60.)

Twelfth

Divide 130 into three parts such that the first (part) is the second (part) divided by 2, and the second (part) is $1\frac{1}{2}$ times the third (part).

The first part is 30, the second part is 60, and the third part is 40.

[1] Scheubel omitted the 28.



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Thirteenth

Three tradesmen invest 170 gold coins (aurei) in a copartnership. The first (man) expresses in the contract that his money shall be returned to him in three months, the second (man that his money shall be returned) in 6 months, and the third (man that his money shall be returned) in 8 months. In a certain time this money bears interest amounting to 375 florin. The first (man) receives for his part, including his share and interest, 75 florin; the second (man receives), 200 florin; and the third (man receives), the remaining florin. How much was each man's share, or how much money was collected from each?[1] The result: 60 gold coins (aurei) from the first (man), 80 from the second (man), and 30 from the third (man).

Operation

Time	Gold
3 months	$1x$
6 months	$1\frac{1}{3}x$
8 months	$\frac{1}{2}x$

$$2\tfrac{5}{6}x = 170$$
$$x = 60$$

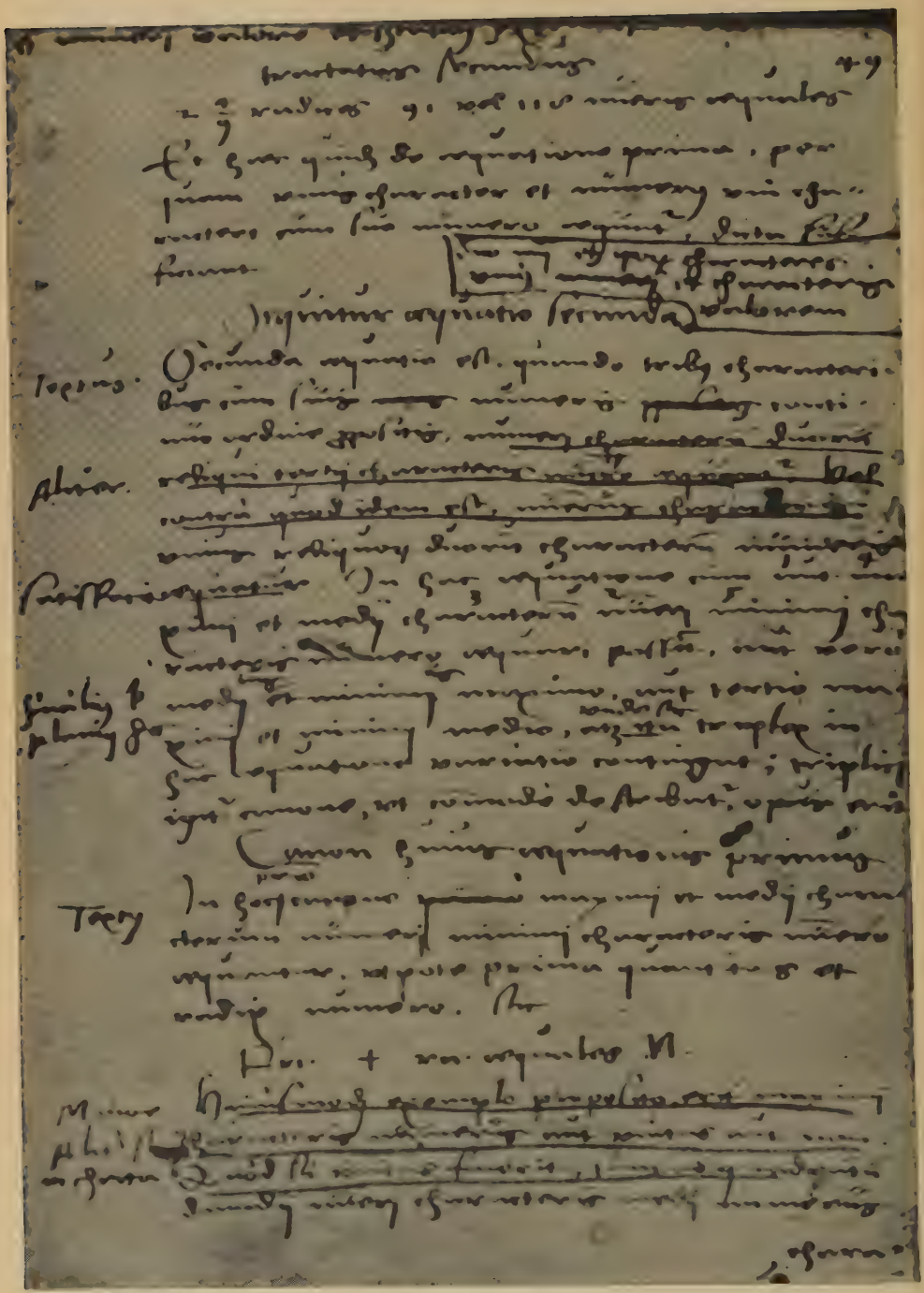
Fourteenth

Divide 91 or 118 into four parts which are in the ratio of one and one half, (that ratio) doubled, and then in the ratio of nine fourths.

The results for the four parts are:

91 ..... $37\frac{5}{22}$, $24\frac{9}{11}$, $12\frac{9}{22}$, $16\frac{6}{11}$.
118 ..... $48\frac{3}{11}$, $32\frac{2}{11}$, $16\frac{1}{11}$, $21\frac{5}{11}$.

[1] This problem is included in Scheubel's algebra of 1551. Recorde, *Whetstone of witte* (1557) fol. I i 3, *r*, gives a problem concerning "A question of money."



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Operation

$$x, \tfrac{2}{3}x, \tfrac{1}{3}x, \tfrac{4}{9}x; 2\tfrac{4}{9}x = 91 \text{ or } 118$$

These (examples) are sufficient to illustrate the first equation, which consists of one character and its number equal to another character with its number.

Concerning the Second Equation[1]

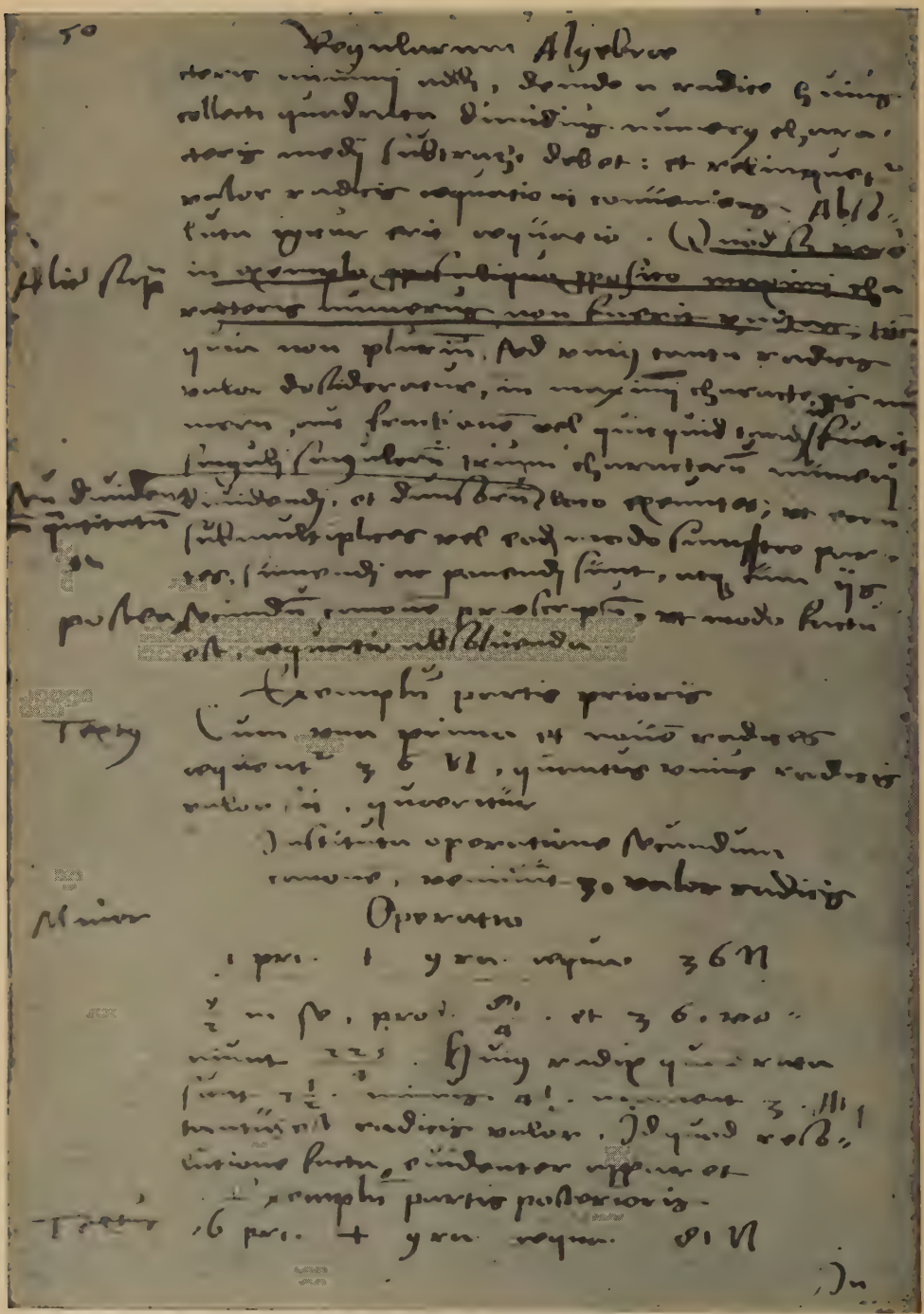
In the second equation three successive characters with their numbers are given, and the numbers (representing) two characters are equal to the number (representing) the third (character), or, vice versa, the number (representing) one character equals the numbers (representing) the other two characters. In this equation the numbers (representing) the highest and the middle characters equal the number (representing) the lowest (character), or (the numbers representing) the middle and the lowest (characters) equal (the number) representing the highest (character), or, as a final possibility, (the numbers representing) the highest and lowest (characters) equal the (number representing) the middle (character). Since there are three types of this equation, the rule must consist of three parts to describe (the different types).

The First Rule for the Second Equation

Here the numbers (representing) the highest and middle characters are equal to the number (representing) the lowest character, such as $x^2 + x =$ a number, $(x^2 + x = N)$.

In an example of this type the number of the (highest) character may or may not be unity. If it is unity, then to the square of one half of the number of the middle character should be added the number of the lowest character;

[1] Recorde, *Whetstone of witte*, fol. F f 4, *v*, makes the following interesting comparison: "The second kind of equation, after Scheubelius mind and myne also, is, when one simple nomber Cossike is compared as equal to 2 other simple nombers cossike, of severalle denominations. And in soche equation beyng reduced as is taught before, the roote of those 2 nombers compounded, as in 1 (or rather the value thereof) shall be extracted: as I have before taut. And that roote doeth answere to the question."



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then from the square root of this sum half of the number of the middle character should be subtracted, and the remainder is the value of the root, or a solution of the equation.[1] But if in any given example the number of the highest character is not unity, then, because the value of only one root is wanted, each of the numbers of each of the three characters should be divided by the number of the highest character, whether it is a fraction or a whole number or a decimal, and the resulting quotients, submultiples or fractions, should be solved according to the prescribed rule.

An Illustration of the First Type

If (a number) of the first quantity and nine roots equal thirty-six numbers, ($x^2 + 9x = 36$), find the value of the root.

When the rule is applied, the value of the root is found to be 3.

Operation

$$x^2 + 9x = 36$$

$\frac{9}{2}$ squared $= \frac{81}{4}$ $\qquad \frac{81}{4} + 36 = \frac{225}{4}$

The square root of this is $\frac{15}{2}$ or $7\frac{1}{2}$; $7\frac{1}{2} - 4\frac{1}{2} = 3$, the value of the root. It is clearly seen (from the above solution) how this value is obtained.

An Illustration of the Second Type

$$6x^2 + 9x = 81$$

[1] Euclid, in his *Elements* (II, ii), solves the equivalent of $x^2+ax=a$ and $x^2+ax=b^2$ by completing the geometric square and neglecting negative roots.

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In this example, since the number of the highest character is not unity, but another number, the numbers of the different characters must be divided by it before the second rule can be applied. The result or value of the root is 3. This can be proved.

THE SECOND RULE FOR THE (SECOND) EQUATION[1]

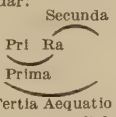
In this second rule (the numbers representing) the middle and lowest characters equal (the number representing) the highest character.

$$x + a \text{ number} = x^2 \; (x + N = x^2)$$

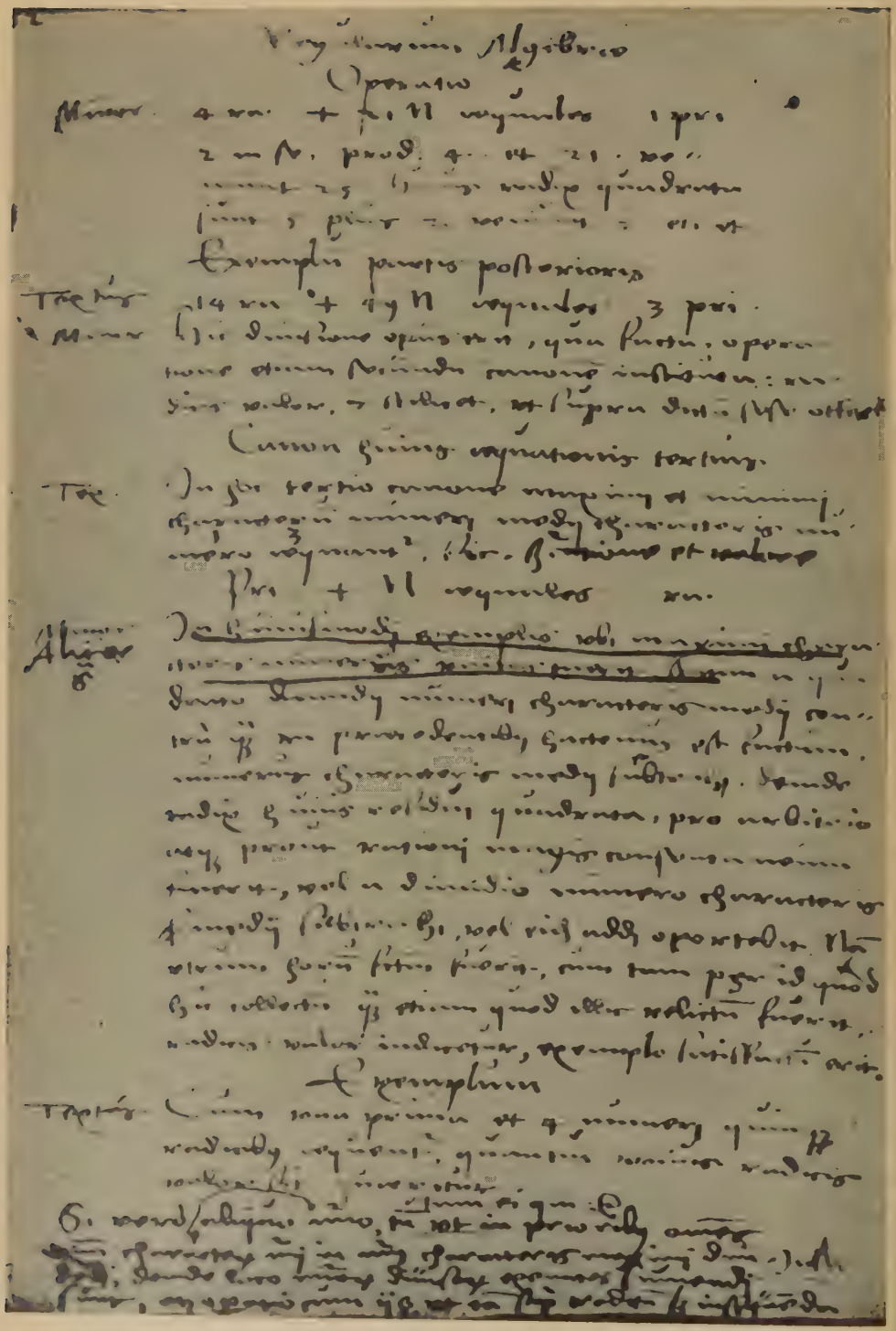
The number of the highest character may or may not be unity. If it is unity, then to the square of $\frac{1}{2}$ of the number of the middle character, as in the preceding rule, the number of the smallest character should be added. Then, when the square root of this sum is added to one half of the number of the middle character, the equation is completed. But if, in an example, the number of the highest character is not unity, then, as in the preceding rule, it should be reduced to unity by division.

If 4 roots and 21 numbers equal a number of the first quantity, $(4x + 21 = x^2)$, find the value of the root.

[1] Treutlein, *Die Coss*, p. 94, discusses Scheubel's representation given on page 19, *r* of his 1551 edition: "Die zur *Aequatio secunda* gehörigen Einzelfälle trennt Scheubel in altherkömmlicher Weise und stellt dies figürlich so dar:



Bei dem *Canon huius aequationis tertius*, nämlich: *Pri+N aequales ra*; d. h. $x^2+b=ax$, zeigt er zwar, dass zwei Lösungen auftreten können, ist aber, wie Stifel's Vorgänger, nicht der Ansicht, dass stets beide zulässig seien."



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Operation

$4x + 21 = x^2$. The square of one-half of four is four; $4 + 21 = 25$; $\sqrt{25} = 5$; $\frac{1}{2}$ of $4 = 2$; $5 + 2 = 7$.

An Example of the Last Type

$$14x + 49 = 3x^2$$

After dividing (each number by 3) and applying the given rule, the value of the root is found to be 7, as in the above (example).

THE THIRD RULE OF THIS EQUATION

In the third rule, the numbers representing the highest and lowest characters are equal to the number (representing) the middle character, as in $x^2 +$ a number $= x$, $(x^2 + N = x)$. In examples of this type, when the number of the highest character is unity, from the square of $\frac{1}{2}$ of the number of the middle character the number of the lowest character is subtracted, and the result of the square root of this remainder and $\frac{1}{2}$ of the number of the middle character are subtracted and then added. These (two) values of the root satisfy the equation.

(But if the number of the highest character is other than unity, then, as in the previous examples, all the characters are to be divided by this number, and the resulting quotients are to be solved according to the above rule).

An Example

Find the value of the root (of the equation) $x^2 + 4 = 5x$. When the given rule

Operatio

Minor 1 pri. + 4 N aequ. 5 re.

$\frac{5}{2}$ in se, prod. $\frac{25}{4}$. minus 4 re.
manet $\frac{9}{4}$. Horum radix quadrata, sunt
$1\frac{1}{2}$. sed $2\frac{1}{2}$. et manet 1. rad pra
minus 4. Vterq; radicis valor, ut partes
tioni satisfaciunt minoris.

Aliud exemplum

Tertio $2\frac{2}{3}$ pri. + $9\frac{1}{3}$ N aequ. 13 re.
Minor Quadratum aequilateri rectangulorum minoris in
maiorem rectangulo aequari, inuenire latus
quadrati quaesiti minoris, et maioris quadrati. $\frac{3}{5}$
vel 4, vtrumq; radicis valor. Quod quaesitum
fuit quaest

Epilogus

Hactenus quidem tria vel sex exempla quinq;
quadratorum aequationum exempla, eorumq; valida
et [illegible] praecipuorum minorum, plurium sunt,
omnia alia ex his generalibus exemplis facilime

Disputantur regulis pro hac quadratorum
aequationum quinquenum quaestionum

Alia exempla alia

Tertia Quadratum [illegible] quadratorum
duo minorum, in maiorem $2\frac{1}{3}$. et 13 eorum
aliorum multiplicatio, producto latus

minus

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is applied, the value of the root is found to be 4.

Operation

$$x^2 + 4 = 5x$$

$(\frac{5}{2})^2 = \frac{25}{4}$; $\frac{25}{4} - 4 = \frac{9}{4}$; $\sqrt{\frac{9}{4}} = 1\frac{1}{2}$; $1\frac{1}{2} + 2\frac{1}{2} = 4$; $1\frac{1}{2}$ from $2\frac{1}{2} = 1$.

Both 4 and 1 are values of the root resulting from trial (inspection).

Another Example

$$2\tfrac{2}{3}\, x^2 + 9\tfrac{1}{3} = 13x.$$

When the numbers of the single characters are divided by the number of the highest character, and the (above) rule is applied, the result is $\frac{7}{8}$ and 4, the two values of the root.

Epilogue

In the same way that the 3 or 6 examples of the second equation have been solved, each according to a special rule, so all other examples of this type may be solved.

Problems and Illustrations of the Second Equation

First

When the sum of two unknown numbers, which are in the ratio of $2\frac{1}{3}$, is added to their product,

Trigonometrica algebraica

Minuendi diuersi, 3i uel $1\,42\frac{1}{2}$ applicandi

minor		minor
7		3
$\text{an. } 23^{s}\ \frac{5}{18} - 1\frac{2}{3}$		$\text{an. } 61^{s}\ \frac{5}{90} - \frac{5}{7}$

Operatio

Minor minor Pendentes.

$1\ \text{an} \quad \frac{3}{7}\ \text{an.} \quad \frac{3}{7}$ primae

Aliter tandem

$\frac{3}{7}$ pri. $+ 1\frac{3}{7}$ an. aequ. 3i uel $1\,42\frac{1}{2}$ N.

Deinde una tertia, uerbi gra

$\text{Una pri.} + 3\frac{1}{3}\ \text{an.}$ aequ. $72\frac{1}{2}$ uel $22\frac{1}{2}$ N

Sunt duo exempla eiusdem primi

aequationis. Nunc duo

Potest in operationibus nostris accidere, minorem minori. Quod ubi fuerit, minor etiam a maiori minori per regulas traditas, quam operatio docet, separemus

Exemplum Secundae Problematis algebraici primi. Inuenire tria numera in proportione Geometrica, quorum primus ad secundum habeat rationem datam, secundus autem ad tertium, ita quidem ut quadratum secundi ductum in summam primi & tertii faciat numerum datum. Sit ratio primi ad secundum, ut 2 ad 3, secundi autem ad tertium, ut 3 ad 5, quadratum uero secundi ductum in primum & tertium faciat 1215. Quaeritur primus, secundus & tertius.

Sint primus x, secundus y, tertius z

$2\,5^{s}\ \frac{1}{2}$ — in 5^{s} septimum 3 dicatur

$5\,6^{s}\ \frac{1}{8}$ — in quintum unus minus tres

Opus

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the result is 31 or $142\frac{1}{2}$.

7 is the greater number and 3 is the smaller number.

The greater (number)	Operation The smaller (number)	The product (of the numbers)
x	$\frac{3}{7}x$	$\frac{3}{7}x^2$

Then, $\frac{3}{7}x^2 + 1\frac{3}{7}x = 31$ or $142\frac{1}{2}$.

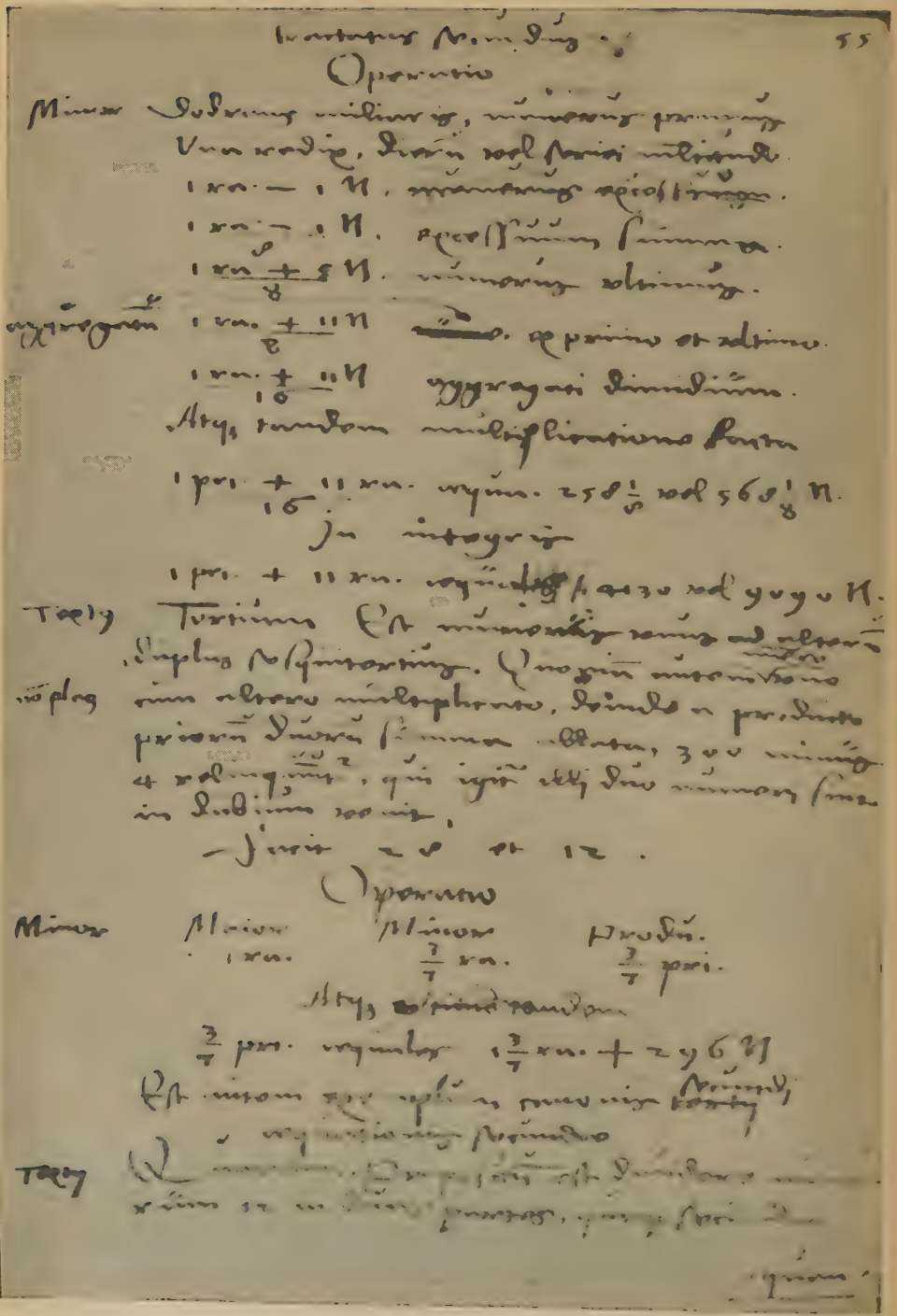
Dividing, $x^2 + 3\frac{1}{3}x = 72\frac{1}{3}$ or $332\frac{1}{2}$.

These two examples illustrate the first rule of the second equation. When in a solution the root, (the unknown or x), is represented as the larger number, then the smaller number, indicated in the solution (in terms of this value), can be found.

Second

Someone travels[1] 12 miles the first day, and every day afterwards he travels $\frac{1}{8}$ mile more than on the preceding day, thus forming an arithmetical progression. He decides to travel $258\frac{1}{8}$ or $568\frac{1}{8}$ miles; in what time can he do it? The result is $258\frac{1}{8}$ miles in 8 weeks and 3 days, or $568\frac{1}{8}$ miles in $\frac{1}{4}$ year less 1 day.

[1] This problem is another illustration of Scheubel's influence predominating in Recorde's algebra: "There is a strannge jorneye appointed to a manne. The first daie he must goe $1\frac{1}{4}$ mile, and every daie after the firste, he must still augemente his jorney, by $\frac{1}{4}$ mile, so that his jorney shall procede by an arithmetical progressione. And he hath to travel for his whole jorney 2955 miles. I demande in what nomber of daies, shall he neade his jorney?", *Whetstone of witte* (1557), fol. I i 4, *r*.



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Operation

12 miles, the first number.

x, the number of days, or series.

$x - 1$, = the number in excess.

$\frac{x-1}{8}$, the sum of the excesses.

$\frac{x+5}{8}$, the last number.

$\frac{x+11}{8}$, the sum of the first and last.

$\frac{x+11}{16}$, half the sum of the first and last.

Multiplying, $\frac{x^2+11x}{16} = 258\frac{1}{8}$ or $568\frac{1}{8}$.

Clearing of fractions, $x^2 + 11x = 4180$ or 9090.

Third

A number is $2\frac{1}{3}$ times another (number). If one number is multiplied by the other (number), the sum of the two numbers subtracted from the product, there remains 300 minus 4. What are the numbers? They are 28 and 2.

Operation

Greater	Less	Product
x	$\frac{3}{7}x$	$\frac{3}{7}x^2$

$\frac{3}{7}x^2 = 1\frac{3}{7}x + 296.$

These are examples of the second rule of the second equation.

Fourth

The number 12 is divided into two parts. When the sum of the numbers is added to the sum of their squares, and cubes,

Regulæ Algebræ

quantitatem unius numeri primi, et ejus
plus numeris. 85 et hujus, Quæritur
fiant 9 et 3.

Operatio

Minor 1 res. 12 N — 1 res.
1 pri. 144 N + 1 pri. — 24 res.
1 Cub. 1728 N + 36 pri. — 432 res. — 1 C.

Atq; tandem et ultimò in numeros,

1 pri. + 27 N. æquæ. 12 res.

Et unum exemplum numeris tertij
æquatur nihilo.

Quinque. Dimidium 11 in se, quo-
rum quadratum simul 65 vel 73 horum
Quæritur utrum.

Fiant 4 et [illegible] vel [illegible] et 8

Operatio

Minor 1 res. 11 N — 1 res.
1 pri. 121 N + 1 pri — 22 res.

Atq; ultimò unus in nihilo

1 pri. + 28 vel 24 N. æquæ. 11 res.

Unum duo exemplum numeris tertij æquæ-
tur nihilo

Septem. Sunt duo numeri in rectum
tripla. Quorum unus in ultimo multi-
plicus, de producto deinde minimo numero
subtracto, 216, vel 15 + √243 relinquunt
quis igitur 9, duo tripla rectionis numeri,
sint, quæritur

Sint quæsiti ad rectum

primus 27 . 9 .
Numerus 6 + √27 . 2 + √3

Ope-

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the result is 858. The parts are 9 and 3.

Operation

x	$12 - x$
x^2	$144 + x^2 - 24x$
x^3	$1728 + 36x^2 - 432x - x^3$

Adding and reducing to lowest terms, $x^2 + 27 = 12x$.

This is an illustration of the third rule of the second equation.

Fifth

Divide 11 into two parts such that the sum of their squares equals 65 or 73. The results are 4 and 7, or 3 and 8.

Operation

x	$11 - x$
x^2	$121 + x^2 - 22x$

Reducing, $x^2 + 28$ or $24 = 11x$

This is an illustration of the third rule of the second equation.

Sixth

One of two numbers is three times the other. From the product of these two numbers subtract the larger number. The remainder is 216 or $15 + \sqrt{243}$. Find the numbers. They are 27 and 9 or $(6 + \sqrt{27})$ and $(2 + \sqrt{3}.)$

Operation of the First Part

Tractatus Secundus

Maior Operatio partis prioris

Maior minor Vel Minor ma-

1 ra. $\frac{1}{3}$ ra. 1 ra. 3 ra.

Atq; ultimo tunc

1 pra. aequa. 3 ra. + 6 q 8 N

Vel 1 pra. aequa. 1 ra. + 7 2 N

Operatio partis posterioris

1 ra maior $\frac{1}{3}$ ra. minor nu.

Atq; tunc multiplicatione et subtractione facta

$\frac{1}{3}$ pra. — 1 ra. aequa. 15 + $\sqrt{2}$ 43 N

Vel alibi aequabitur

$\frac{1}{3}$ pra. aequa. 1 ra. plus 15 + $\sqrt{243}$ N.

Et inuenta [illegible] tunc

Una pra. aequa. 3 ra. plus 45 + $\sqrt{2187}$ N.

Sunt autem anguli numeri Secundi aequationis secundae. Operationem igitur ab ipsi prae cipiemus, nisi tantum radicem radicum ad superficiem, ipsius aeq. reprobatis.

Septimus Secundae 6 in duo, et unum ex eis multiplicato [illegible], quorum fiet 3 + $\sqrt{2}$ et 3 — $\sqrt{2}$.

Hactenus de aequatione Secunda.

Sequitur aequatio tertia

Tertia aequatio est, quando tribus aequalibus tribus cum suis numeris, non aequalium ordine [illegible], sed ab singulis inter eos proportionalibus unus et unus, binj et binj vel terni et terni [illegible]

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Greater	Smaller	or Smaller	Greater
x	$\frac{1}{3}x$	x	$3x$

Then, $x^2 = 3x + 648$

$x^4 = 1x + 72$

Operation of the Last Part

The greater number = x. The lesser number = $\frac{1}{3}x$.
Multiplying and subtracting, $\frac{1}{3}x^2 - x = 15 + \sqrt{243}$.
Adding equals, $\frac{1}{3}x^2 = x + 15 + 243$.
Reducing, $x^2 = 3x + 45 + \sqrt{2187}$.

These are examples of the second rule of the second equation. When the indicated operations are performed the result will give the above value (for the root).

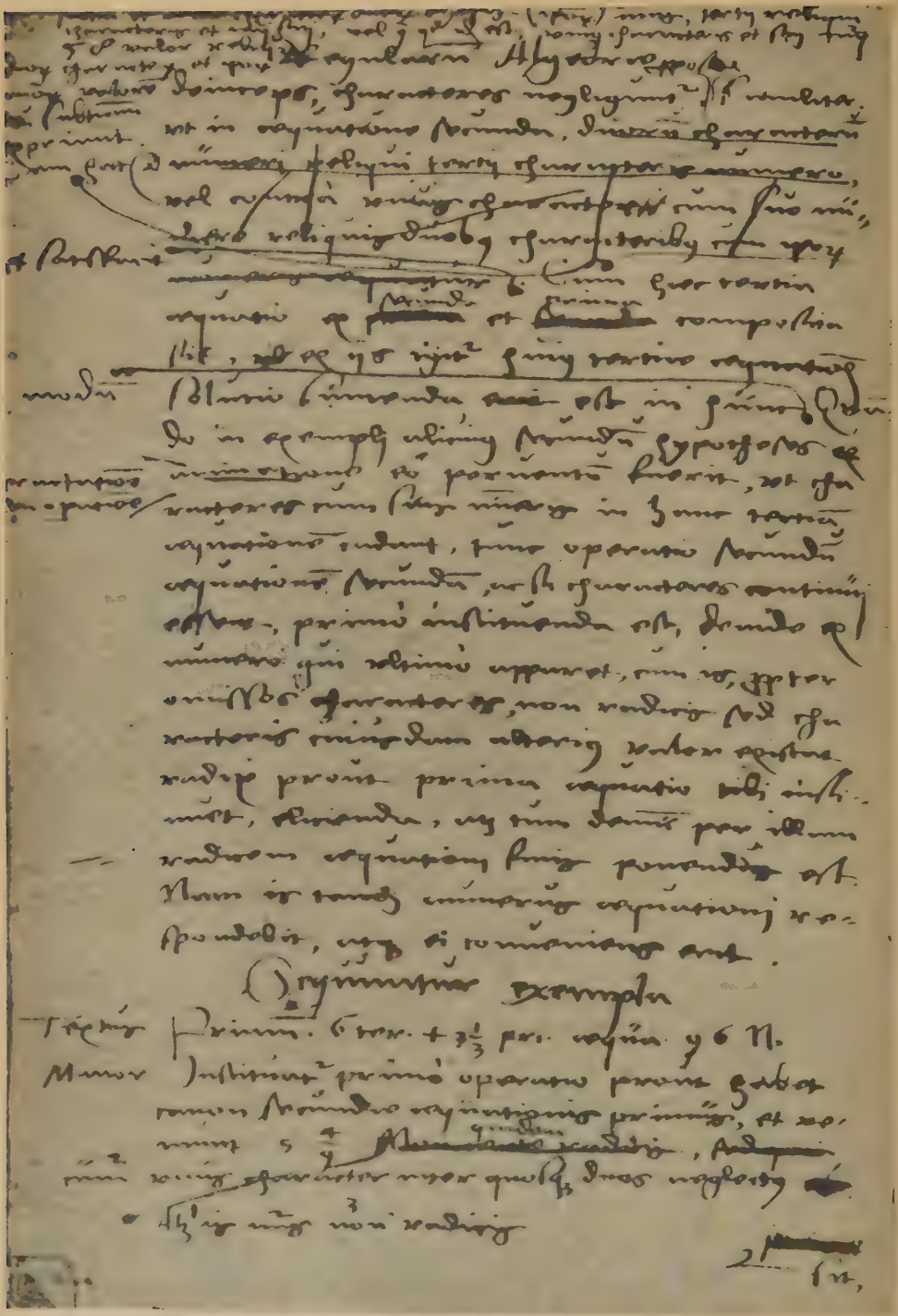
Seventh

Divide 6 into two parts such that their product is 7.
$3 + \sqrt{2}$ and $3 - \sqrt{2}$ are the parts.
These illustrations are sufficient for the second equation.

The Third Equation[1]

In the third equation characters with their numbers are not given in successive order, that is, characters are omitted between one (character) and the next in order, such as between the first and second (characters), (between) the second and third (characters),

[1] Recorde, *Whetstone of witte* (1557), fol. F f 3, *r*, does not consider this equation necessary: "Now will I shewe you the varities of equations, taught by Scheubelius, bicause you maie perceive, how thei bee conteined in those 2 formes, named by me. And for the manyfolde varities that some other doe teache, I accoumpte it but an idle bablyng, or (to speake moare favourably of them) an unnecessary distinction;" on fol. F f 4, *r*, he remarks concerning omitted characters: "But if there bee any denomination omitted, between those 2, . . . ever more you shall extracte the root of your laste number."



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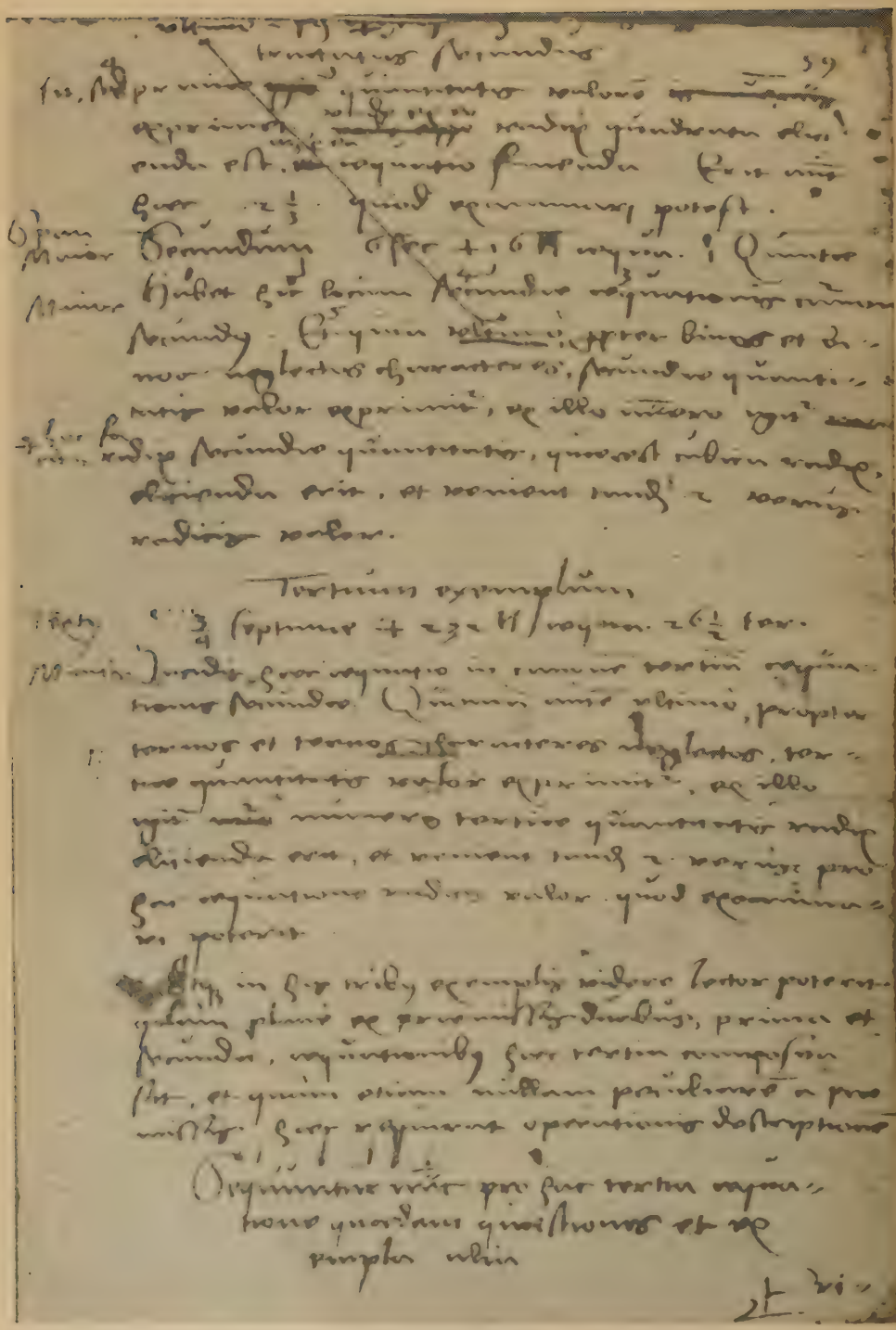
and so on successively, just as they were illustrated in the second equation.

When the third equation is composed of the first and second equations, its solution is an application of the (rules for) the solution of these two (equations). If in any solution, according to the given hypothesis, characters with their numbers are continuous in the third equation, then the solution (of this equation) is an application (of the rules) of the second equation from which (the characters) originated. When characters are omitted, the value of the root, taken from the equation from which (the characters) originated, is not the real value, but it is (the value) of another character whose root must be found and substituted (in the equation) to satisfy (the equation).

Examples

$$6x^4 + 3\ \tfrac{1}{3}x^2 = 96$$

When the first rule of the second equation is applied, the result is $5\frac{4}{9}$, a value of the first quantity, (and this is not the value of the root, since one character between the first and second term is omitted).



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When the square root is extracted, (the solution) of the equation is completed. The result, $2\frac{1}{3}$, can be proved.

Second

$$6x^3 + 16 = x^6$$

This (equation) is an (application) of the second rule of the second equation. Since two characters are omitted, the value of the second quantity, or cube root, is expressed. When the cube root is extracted, the result is the real value of the root.

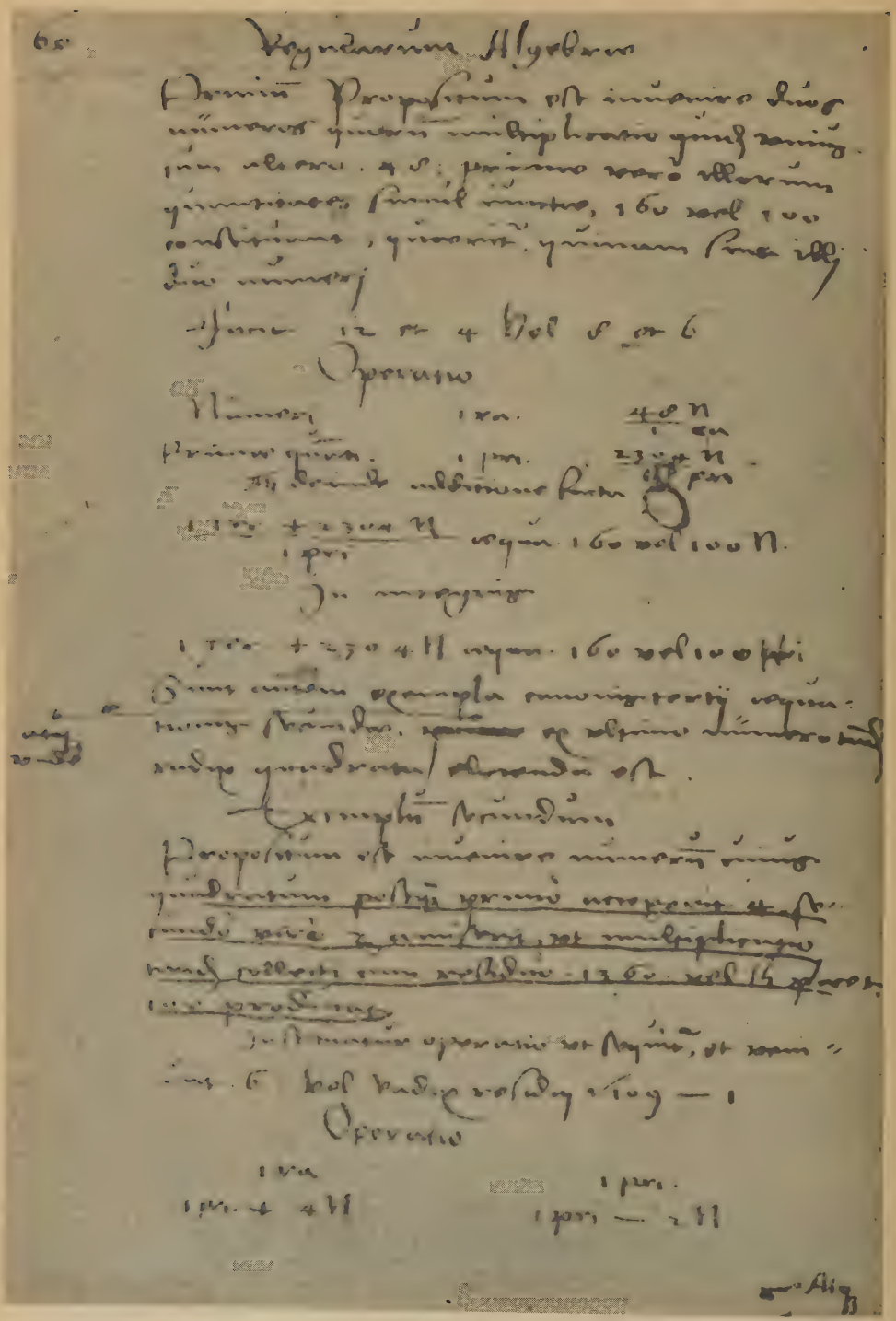
Third

$$\tfrac{3}{4}\, x^8 + 232 = 25\tfrac{1}{2} x^4$$

This is an application of the third rule of the second equation. Since three characters are omitted, the value of the third quantity (or fourth root), is expressed. When the root of the third quantity (or the fourth root) is extracted, the result is the real value of the root of this equation. This can be proved.

From these three examples the reader can plainly see how the third equation is composed of the two previous (equations), the first and second equations; therefore no special explanation of the solution is necessary.

Illustrative Questions and Answers



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First

Find two numbers whose product is 48, and the sum of whose squares is 160 or 100. They are 12 and 4, or 8 and 6.

Operation

Numbers, x, $\frac{48}{x}$

The squares, x^2, $\frac{2304}{x^2}$

Adding, $x^4 + \frac{2304}{x^2} = 160$ or 100.

Clearing, $x^4 + 2304 = 160x^2$, or $100x^2$

These are examples of the third rule of the second equation, in which the square root of the first quantity must be extracted.

Second

Find a number such that the product of its square increased by 4 and its square diminished by 2 will give 1360 or 100. The solution gives as a result, 6, or the root of the residual $\sqrt{109} - 1$.

Operation

x	x^2
$x^2 + 4$	$x^2 - 2$

tractatus Arundus

Atq; multiplicentur hæc, quemadmodum oblatum

1 Tert. + 2 pri. aequa. 13 6 ℵ

1 Tert. + 2 pri. aequa. 1 0 ℵ

Quæ cum in Arundel aequationes quamplurimas prius, quam Arundo ullius præceptionem aperuerit instituenda est. Demum cum operationis absolutum 36. reel √190 — 1. primus quantitatis radix, Hobbes, Radix ergo radicis est ipse numerus 6. reel Radix radicis √190 — 1 Quod de reliquis probare potest

Aequationes cum pro tribus aequationibus, perturbato tamen ordine, exemplis ubique

Primum Propositum est Arundeo 13 in duos partes ita, ut rectangulum in altera diuisa, aequetur simul numero quadrato, qui fiat ex multiplicatis, in quem numerus producat

sint 10 et 3

Operatur

Numeri 1 rad et 13 ℵ — 1 qu.

Primus $\frac{1 \text{ qu.}}{13 \text{ ℵ} - 1 \text{ rad.}}$ et $\frac{13 \text{ ℵ} - 1 \text{ qu.}}{1 \text{ rad.}}$

Secundus simul $\frac{169 \text{ ℵ} - 2 \text{ pri.} - 26 \text{ qu.}}{13 \text{ rad.} - 1 \text{ pri.}}$

Atq; multiplicentur cum 5. facta radicum producunt

$\frac{845 \text{ ℵ} + 10 \text{ pri.} - 130 \text{ qu.}}{13 \text{ rad.} - 1 \text{ pri.}}$ aequa. $18\frac{1}{0}$ ℵ.

Da

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Multiplying, $x^4 + 2x^2 = 1368$
$x^4 + 2x^2 = 108$

These examples, illustrating the first rule of the second equation, are to be solved according to the above rule. After performing all of the operations, the result is 36 or $\sqrt{109} - 1$. The square root, or value, is 6, or the root of the residual $\sqrt{109} - 1$. This can be proved.

Other Examples Illustrating All Three Equations in Mixed Order

First

Divide 13 into two parts such that when either (part) is divided by the other and the sum of the two parts is multiplied by 5, the product is $18\frac{1}{6}$. The parts are 10 and 3.

Operation

Numbers, x, $13 - x$.

Divisions, $\frac{x}{13 - x}, \frac{13 - x}{x}$

Adding, $\frac{169 - 2x^2 - 26x}{13x - x^2}$

Multiplying, $\frac{845 + 10x^2 - 130x}{13x - x^2} = 18\frac{1}{6}$

Regulæ Algebræ

In integris tandem et relinquo.

Dum proi. + 30 N vapua. 13 rem:

Secundum. Propositum est dividere 10 vel 13 in duas partes, quarum minoris qua ducatur in se hinc, quinto numero producto, 10 vel 13. cum sua parte minore multiplicatus producit

	minor quæ	minor per.
10 —	$\sqrt{125} - 5$	$15 - \sqrt{125}$
13 —	$\sqrt{211\frac{1}{4}} - 6\frac{1}{2}$	$14\frac{1}{2} - \sqrt{211\frac{1}{4}}$

Solutio est facilis

Tertium. Est numerus in duas partes dividendus. Et quarum minoris quadratum unum est quinto ipse dividendus cum minore parte multiplicato producit, quinto ipsa tota numerus minor non parte hinc, cum minore parte se hyperbole $\sqrt{100} - 6$ æquales, quæritur.

Totus nu: minor pars

Divisit 12 et $10 - \sqrt{100}$

Quartum. Quinto mos. primus quadratum, quinto solidus debet 32 unitates. Secundus vero

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Clearing, $x^2 + 30 = 13x$

Second

Divide 10 or 13 into two parts. The square of the greater (part) is as much as the number 10 or 13 multiplied by the smaller part.

If the number is 10, the greater part is $\sqrt{125} - 5$, the smaller part is $15 - \sqrt{125}$.

If the number is 13, the greater part is $\sqrt{211\frac{1}{4}} - 6\frac{1}{2}$; the smaller part is $19\frac{1}{2} - \sqrt{211\frac{1}{4}}$.

The solution is simple.

Third

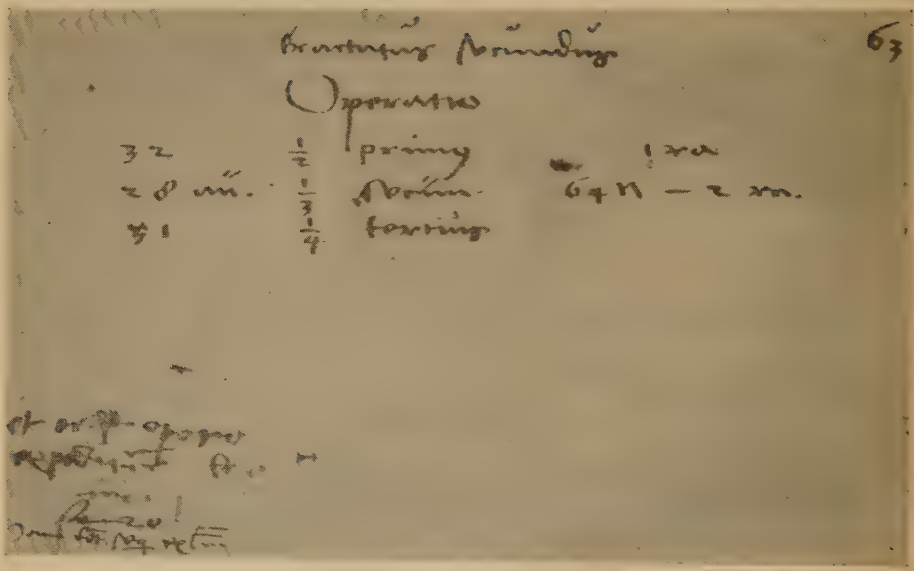
A number is divided into two parts. The square of the greater part is the number itself multiplied by the smaller (part). What will be the whole number and what will be the smaller part if the greater part is by hypothesis $\sqrt{180} - 6$?

The number is 12 and the smaller part is $18 - \sqrt{180}$.

Fourth

Three (men) buy (some) land; the first (man) pays for his share 32 aurei, the second[1]. . . .

[1] Scheubel probably intended to complete this problem or to substitute another problem before his manuscript was published.



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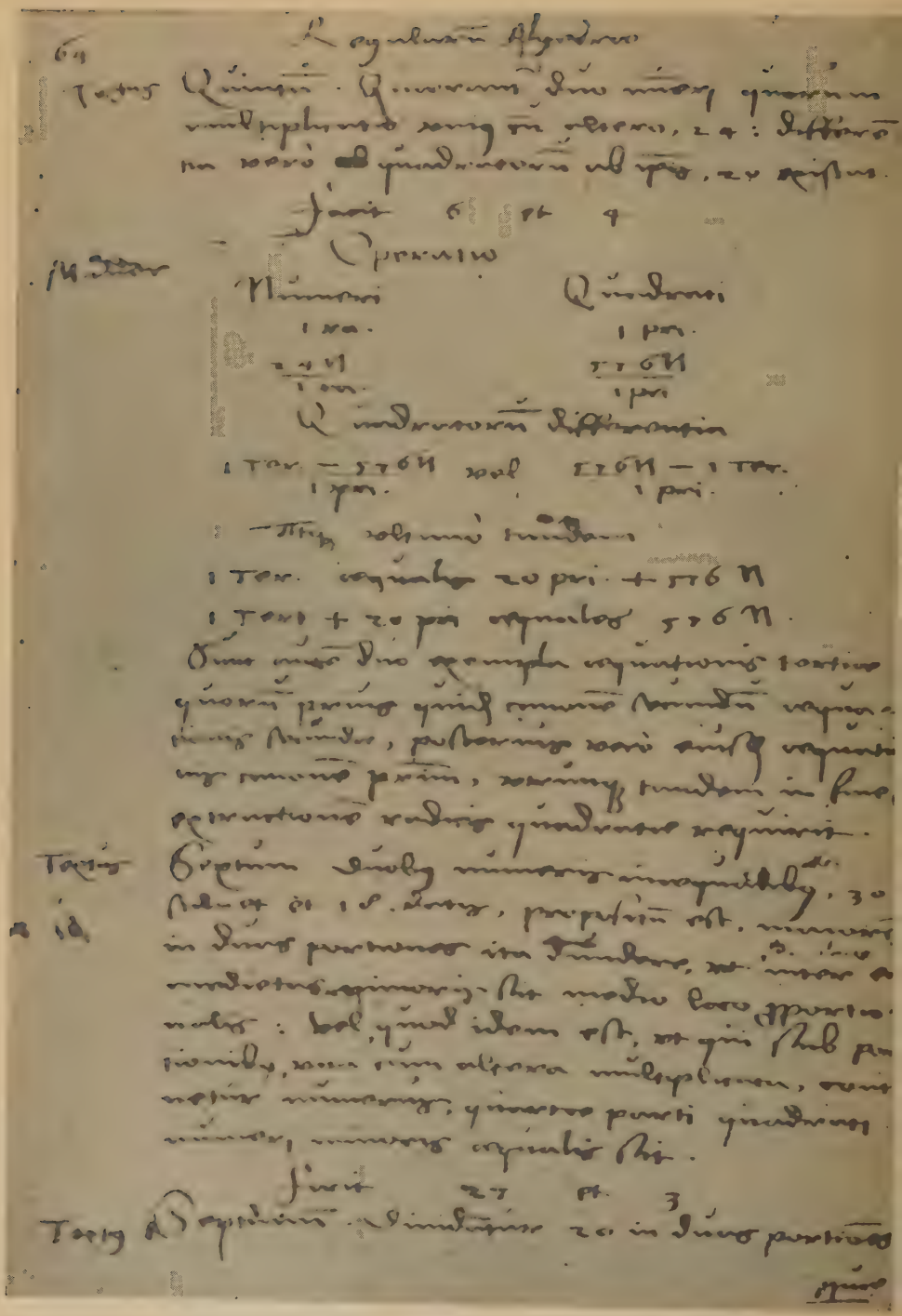
Operation

$32\frac{1}{2}$ first x
$28\frac{1}{3}$ second $64 - 3x$
$31\frac{1}{4}$ third



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Fifth

Find two numbers whose product is 24 and the difference of whose squares is 20. The numbers are 6 and 4.

Operation

The numbers, x, $\frac{24}{x}$

Their squares, x^2, $\frac{576}{x^2}$

The difference of their squares, $\frac{x^4 - 576}{x^2}$, or $\frac{576 - x^4}{x^2}$

Then $x^4 = 20x^2 + 576$

$x^4 - 20x^2 = 576$

These two examples are illustrations of the third equation. In the first (example) apply the second rule of the second equation, and in the last (example) apply the first rule of the second equation. In both (examples) the square root must be extracted.

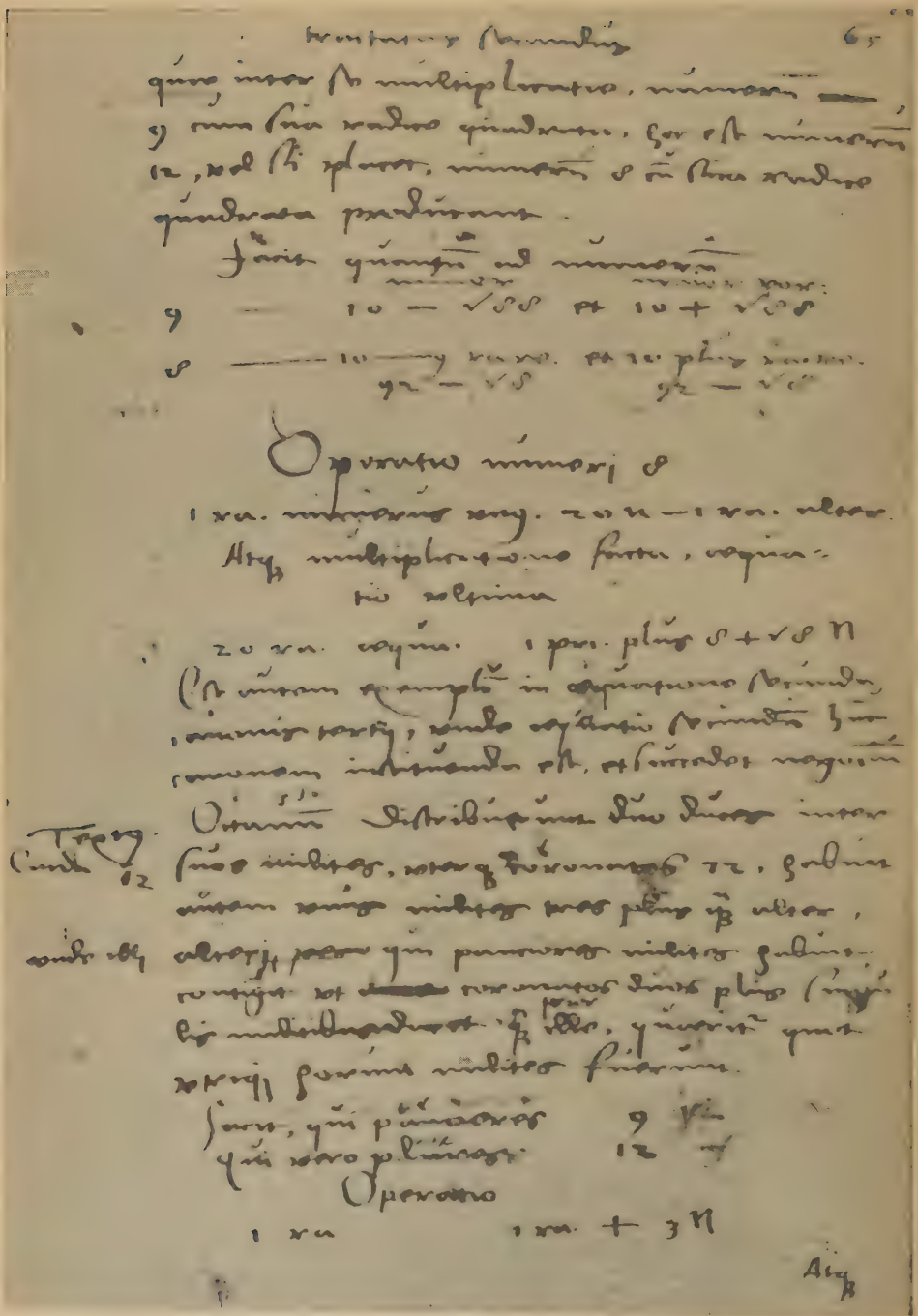
Sixth

Of two unequal numbers, 30 and 18, the greater number is to be divided into two parts whose product will be one fourth the square of the smaller number.

The parts are 27 and 3.

Seventh

Divide 20 into two parts



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whose product is either 9 plus the square root of 9 (or 12), or 8 plus the square root of 8.

The smaller part	The larger part
If the product is $9 + \sqrt{9}$ (or 12),	
$10 - \sqrt{88}$ and	$10 + \sqrt{88}$
If the product is $8 + \sqrt{8}$,	
$10 - \sqrt{92 - \sqrt{8}}$	$10 + \sqrt{92 - \sqrt{8}}$

Operation of the Number 8

x = one number, $20 - x$ other number.

Multiple, $20x = x^2 + 8 + \sqrt{8}$.

This example can be solved by applying the third rule of the second equation.

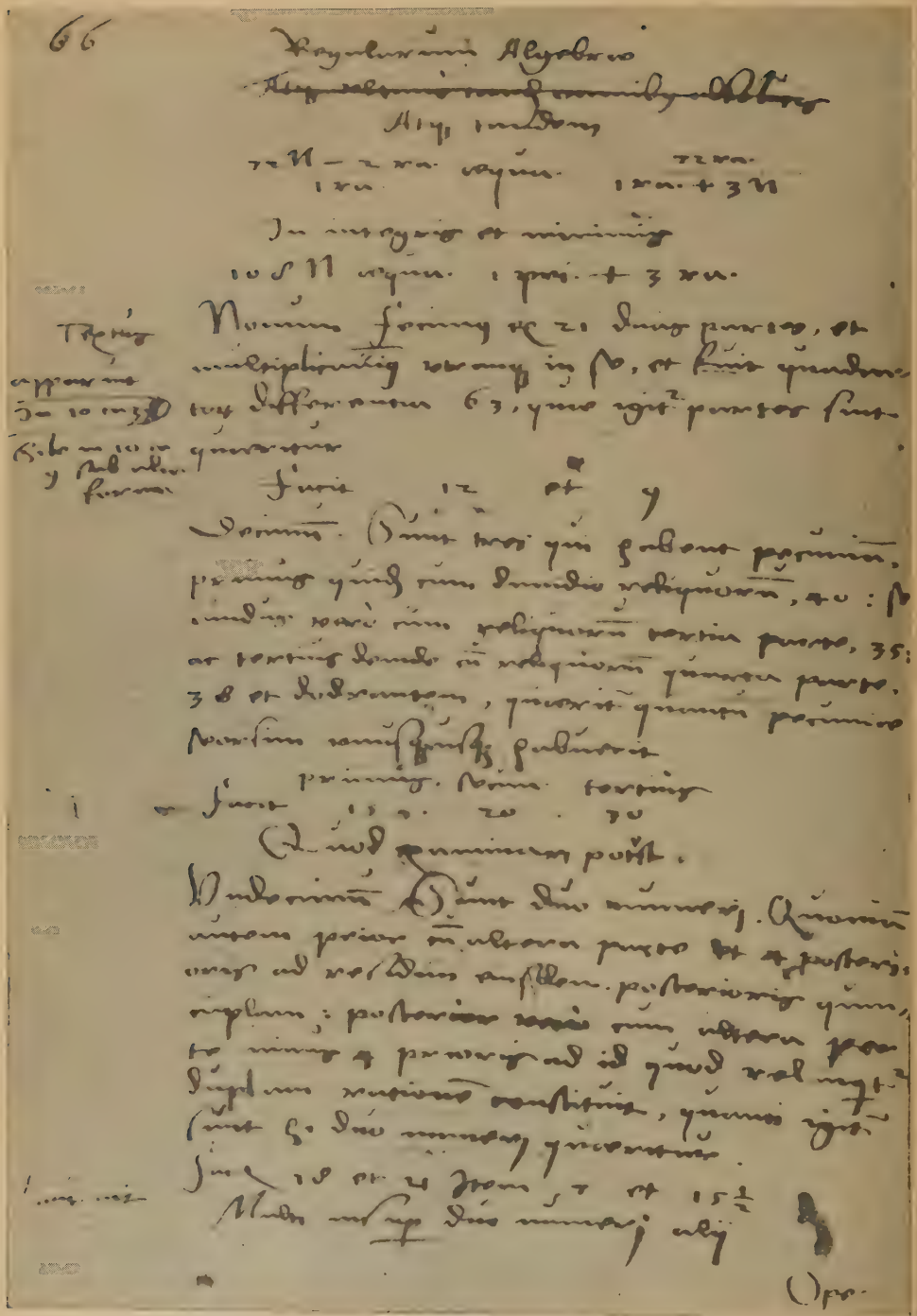
Eighth

Each of two leaders distributes among his soldiers 72 crowns. One leader had three soldiers more than the other. The second leader, who had fewer soldiers, gave two crowns more to each soldier than the first leader did. How many soldiers did each have?

The result, 9 and 12.

Operation

x $x + 3$



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$$\frac{72 - 2x}{x} = \frac{72x}{x + 3}$$

Clearing of fractions, $108 = x^2 + 3x$

Ninth

If 21 is so divided into two parts that the difference between the squares of these parts is 63, what are the parts?

Parts, 19 and 9.

Tenth

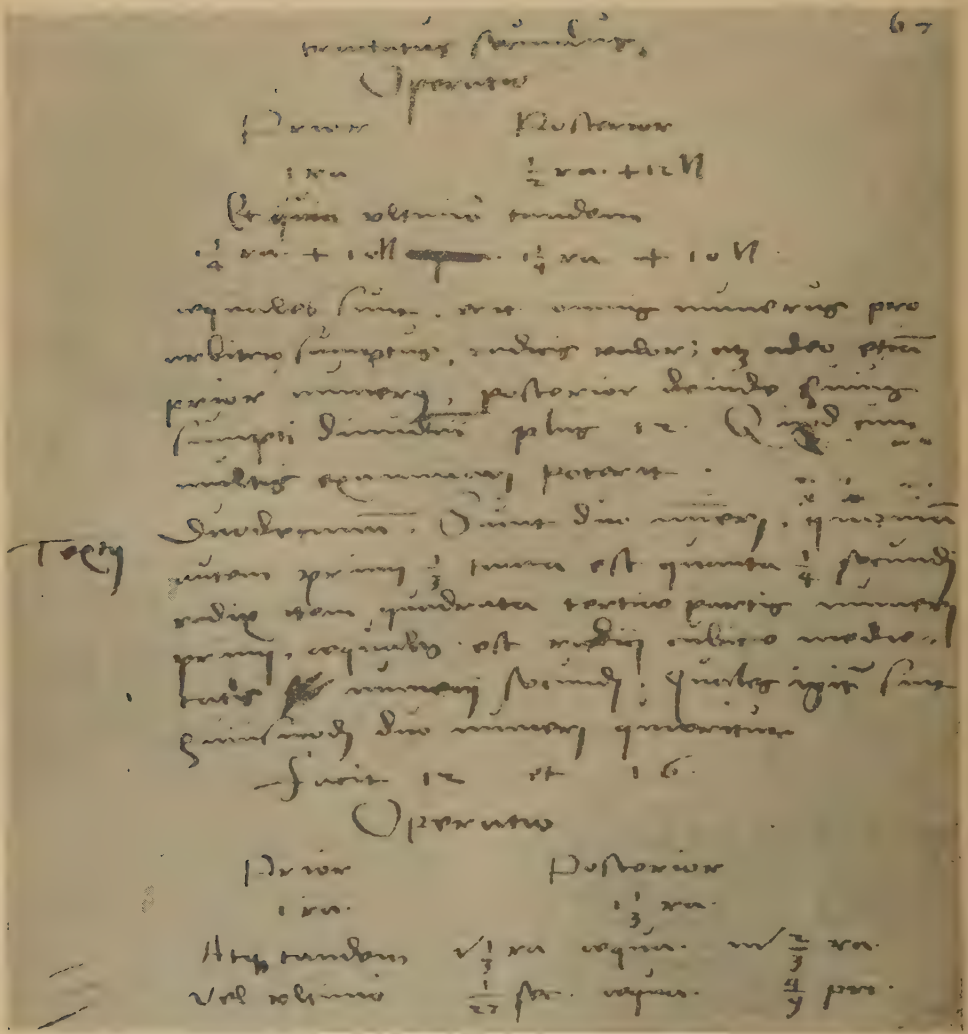
Three men have some money. The first has, with one half as much as the others have, 40 (pieces); the second, with one third as much as the others, 35 (pieces); and the third, with one fourth as much as the others, $38\frac{3}{4}$ (pieces). How many pieces has each man? The first has 15; the second, 20; and the third, 30. This can be proved.

Eleventh[1]

Sunt duo numeri. Quoniam autem prior cum altera parte et 4 posterioris ad residuum eiusdem posterioris quincuplam: posterior vero cum altera parte minus 4 prioris ad id quod relinquitur duplam ratione constituit quanti igitur sunt hi duo numeri quaeritur. Fac 18 et 21. Item 7 et $15\frac{1}{2}$.

Multi insuper duo numeri alii.

[1] Since no translation of this statement agrees with Scheubel's explanation, probably he intended to substitute another problem before the work was printed.



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Operation

First	Second
x	$\frac{x}{2}+12$

Since $1\frac{1}{4}x + 10 = 1\frac{1}{4}x + 10$,[1] any number can be taken arbitrarily for the value of the root, or for the first number. Then the second will be one half of this plus 12. This can be proved with many numbers.

Twelfth

One third of the first of two numbers equals $\frac{1}{4}$ of the second. The square root of one third of the first equals the cube root of half of the second. What are these two numbers? They are 12 and 16.

Operation

First x Second $1\frac{1}{3}x$

Finally $\sqrt{\frac{1}{3}}x = \sqrt[3]{\frac{2}{3}}x$ or $\frac{1}{27}x^3 = \frac{4}{9}x^2$

[1] The solution does not agree with the conditions; however, at the close of his *Algebra* (1551) his solutions of such Greek Epigrams are correct:

"De augeae Armentis, Quotnam boves fuerint. Augeam interrogavit generosus Hercules, de multitudine armentorum cum ille respondit: Media horum pars, amice, circa fluvium Alpheum pascitur: octava autem circa Saturni collem: Caeturum duodecima, procul hinc iuxta loca Taraxippi extrema: at vigesima eorum pars, circa Elidem pascitur: trigesimam vero in Arcadia ego reliqui: reliqua autem, quinquaginta numero armenta, videas ipse. Quaritur.

Facit 20

Summa partium una cum 50 bovibus

Sunt 19/24 ra $+50N$ aequa radici positae. Est prima aequatio, atque radicis valor, armentorum scilicet numerus, 240.

Porro quot nunc in singulis locis vagentur boves, quiuis ex positionis solutione facile cognoscet."

LATIN GLOSSARY

ABBREVIATIONS

adj.,	adjective.	demonstr.,	demonstrative.	m.,	masculine noun.
adv.,	adverb.	f.,	feminine noun.	neut. or n.,	neuter noun.
comp.,	comparative.	indef.,	indefinite.	num.,	numeral.
conj.,	conjunction.	indecl.,	indeclinable.	prep.,	preposition.

This study of the Latinity is not intended to be complete because of the limits of this volume. In indicating the first or unusual use of each word, reference is made first to the page and then to the line in the original text.

absolute, adv., entirely, 15, 22.
absolvo, v., express, declare, 10, 30.
ac., conj., and, 1, 1.
accedo, v., to be added, 1, 5.
addicio, f., addition, 3, 20.
addo, v., add, 11, 14.
adduco, v., complete, take, 22, 5.
adhuc, adv., further, in addition, 12, 1.
adicio, v., add to, 45, 16.
adiectus, adj., added, positive, folio, 61, 1.
admonitio, f., suggestion, 8, 20.
adscribo, v., add or ascribe, 7, 12.
aequalis, adj., equal, 5, 16.
aequalitas, f., the equal, 45, 16.
aequatio, f., equation, equality, 1, 8.
aequus, adj., equal, 45, 25.
affirmatus, adj., positive, 32, 8.
aggregatum, m., sum, 8, 1.
ago, v., operate, 24, 25.
algebra, f., algebra, 1, 1.
algorithmus, m., algorism (algorithm), 3, 2.
aliqui, aliqua, aliquod, indef., pron. adj., some, any, 5, 15.
aliud, adj., another, 5, 6.
ambo, adj., both, 1, 10.
amicio, v., contain, 46, 11.
animadverto, v., consider, 6, 8.
appareo, v., appear, be evident, 50, 29.
appello, v., name, call, 15, 27.
appendix, f., supplement, 29, 17.
aptissimus, adj., proper, useful, 1, 12.
arbitrario (arbitrio), adv., with certainty, 67, 7.
arithmetica, f., arithmetic, 14, 15.
arithmeticus, adj., arithmetical, 54, 20.
ars, f., science, 3, 27.
ascendo, v., ascend, 17, 9.
ascribo, v., apply, ascribe, add to, 7, 19.
assigno, v., mark out, indicate, 20, 13.
assumo, v., take, extract, 20, 15.
astronomia, f., astronomy, 4, 10.
attributum, v., give, 3, 27.
aureus, m., gold coin, 48, 4.
aufero, v., with ex., take away, 5, 29.
binarium, neut., two, 18, 13.
binarius, adj. (consisting of) two, 18, 13.
binominis, adj., binomial, 32, 24.
bis, adv., twice, 18, 20.
bonus, adj., successful, valuable, 31, 17.
breuis, adj., short, 1, 1.
cado, v., fall upon, happen, belong, 43, 11.
calculus, m., calculation, 21, 1
canon, m., rule, canon, 41, 16.
caput, n., chapter, 3, 1.

castra, n., camp, 45, 1.
certe, adv., certainly, of course, 6, 8.
certus, adj., definite, 15, 18.
ceterus, adj., other, 24, 25.
charactereris, n., character or mark, 3, 3.
coaceruatio, v., collect, represent, 17, 20.
cognitus, adj., known, 29, 20.
cognosco, v., understand, recognize, 6, 7.
colligo, v., write together, combine, sum up, 4, 17.
commodum, adv., conveniently, 22, 27.
communis, adj., common, ordinary, 4, 8.
compendium, n., abridgment, 20, 11.
compono, v., compose, make, 58, 7.
comprehendo, v., include, 2, 3.
condedo, v., grant, vouchsafe, 41, 3.
conditio, condition, manner, 17, 5.
confusio, f., confusion, 24, 18.
coniungo, v., connect, join, add, 27, 19.
conscriptus, v., to write, 1, 9.
consequor, v., follow, result, 39, 22.
constituo, v., constitute, make, 45, 24.
consueo, v., accustom, to be used to 4, 15.
contineo, v., contain, 18, 25.
contingo, v., happen, attain, 10, 13.
continuo, v., continue, 58, 14.
contra, adv., opposite, otherwise, 6, 9.
contraho, v., to contract, 48, 3.
contrario, adv., on the other hand, 36, 17.
conueniens, adj., suitable, 50, 14.
copulo, v., join together, 37, 21.
coronatus, m., crown, 65, 21.
cubicus, adj., cubic, 24, 37.
curo, v., administer, take care of, 35, 4.
debeo, v., ought, should, 3, 29.
debilis, adj., weak, lesser, 43, 3.
debitum, adj., proper, 4, 19.
decimus, adj., tenth, 16, 10.
declaro, v., demonstrate, 2, 8.
definitio, f., definition, 15, 15.
deinde, adv., then, 3, 19.
deinceps, adv., successively, 16, 10.
demonstratio, f., demonstration, 1, 2.
denarius, m., unit, 3, 8.
denomino, v., indicate, 3, 4.
denumerationes, list, denomination, 30, 3.
describo, v., describe, 37, 18.
descripcio (tio), f., description, 1, 2.
desidero, v., desire, want, 15, 18.
desum, v., to be left, 6, 7.
descripantia, f., dissimilarity, 16, 5.
deuinco, v., surpass, exceed, 6, 7.
dexter, adj., on the right, 17, 9.
dico, v., name, say, 20, 20.
differentia, f., difference, 64, 2.
difficilis, adj., difficult, 6, 8.
dimidium, neut., half, 55, 8.
diminutio, f., decrease, abatement, 3, 21.
directus, adj., simple, direct, 33, 1.
discribo, v., divide, 65, 17.
diuersitas, f., difference, 5, 14.
diuersus, adj., different, 30, 1.
diuidendus, m., dividend, 56, 27.
diuido; v., divide, 10, 19.
diuisio, f., division, 9, 13.
diuisor, m., divisor, 9, 15.
dodranus, adj., three-fourths, 55, 2.
dragma, unit, 3, 8.
duco, v., draw, 4, 16.
duplatio, f., doubling, 18, 12.
duplex, adj., twofold, double, 31, 22.
duplo, v., double, 19, 18.
duplum, neut., double, 19, 19.
dux, m., general, 45, 1.
eadem, adv., similarly, in same way, 15, 27.
edo, v., publish, bring forth, 1, 12.
effero, v., express, produce, 11, 1.
efficio, v., complete, produce, 35, 9.
elicio, v., draw forth, solve, take square root, 28, 28.
emendatus, adj., corrected, 1, 11.
enunciatio, f., notation, 4, 10.
enuncio, v., to express, 3, 22.

epilogus, m., epilogue, 53, 16.
equo, v., equal, 49, 13.
et, conj., and, plus, 1, 2.
euidenter, adv., manifestly, 50, 29.
examen, neut., testing, 42, 5.
examino, v., test, prove, 15, 1.
excessus, m., excess, 5, 19.
exemplum, neut., example, illustration, 1, 13.
exeo, v., come out, with per, 9, 23.
exercitium, neut., army, 45, 2.
exigo, v., require, demand, 16, 2.
existo (exsisto), v., to be, exist, 5, 18.
expedite, adv., readily, quickly, 20, 16.
expeto, v., desire, want, 15, 24.
expositio, f., explanation, exposition, 21, 14.
exprimo, v., express, show the form of, determine, 3, 24.
extra, prep., with acc., in addition to, 24, 19.
extractio, extraction, 64, 20.
extraho, v., extract, 37, 10.
facilis, adv., easy, 4, 11.
facio, v., make, 26, 13.
figura, f., figure, 45, 3.
finio, v., terminate, complete, 42, 5.
fractio, f., fraction, 10, 20.
fractus, adj., fractional, 23, 15.
fundamentum, neut., foundations, 2, 4.
generalis, adj., general, 42, 15.
genero, v., generate, produce, 17, 3.
genus, neut., class species, kind, 4, 7.
gratia, adv., for sake of, in regard to, 6, 25.
gratia, f., favor, 1, 15.
habeo, v., have, constitute, 3, 3. contain, 5, 13.
hactenus, adv., thus, and so, 5, 12.
hic, adv., here, at this time, 6, 2.
hypothesis, f., hypothesis, 42, 4.
iccirco, adv., therefore, 15, 25.
ideo, adv., on that account, 6, 3.
igitur, conj., therefore, 10, 16.
ignotus, adj., unknown, 41, 27.
illic, adv., there, 22, 26.
illustro, v., illustrate, 1, 13.
imperium, n., command, 45, 1.
in, prep., in; with verbs of division or multiplication, by, 10, 5.
inaequalis, adj., unequal, 5, 18.
inclyta (inclitus), adj., illustrious, 1, 3.
indico, v., indicate, 17, 24.
inferius, adj., lower, 39, 23.
infra, adv., and prep., below, less than, 4, 3.
initium, n., beginning, 6, 6.
institutus, find, appear, establish, 18, 7.
integer, adj., integral, whole, 23, 15.
integrum, n., integer, whole number, 11, 22.
intellectus, adj., understood, 22, 11.
intelligentia, f., understanding, 5, 5.
intelligo, v., understand, 22, 11.
inter, prep., between, 7, 29.
inuenio, v., discover, find, 41, 10.
inuentor, m., inventor, 3, 27.
is, ea, id, demonst., that, the very, the same, 5, 13.
lector, m., reader, 8, 22.
liber, m., book, 1, 5.
liceor, v., estimate, explain, value, 16, 14.
licer, adv., although, 15, 22.
linea, f., line, 4, 15.
locus, m., place, 6, 1.
loquor, v., speak, 41, 28.
lux, m., light, 1, 12.
magnus, comp., maior, maius, adj., great, 10, 14.
maneo, v., remain, abide, 7, 3.
manifestus, adj., evident, 31, 7.
mathematicus, m., mathematician, 1, 6.
medietas, f., half, 25, 26.
medio, v., halve, 37, 15.
medium, neut., middle, half, 46, 24.
medius, adj., middle, mean, half, 46, 28.
mensis, m., month, 48, 4.
mensura, f., measure, degree, 20, 25.
mensuro, v., measure, 4, 9.

methodus, f., rule, method of procedure, 3, 5.
miles, com. soldier, 65, 19.
militaris (miliaris), m., soldier, 55, 2.
millenus, adj., thousand, 45, 2.
minor, minus, comp. of parvus, adj., less, 4, 5.
minutia, f., minute, 4, 10.
modus, m., manner, fashion, 4, 8.
moneta, f., money, 4, 9.
multiplicacio, f., multiplication, 4, 12.
multiplico, v., multiply, 8, 14.
multus, adj., many, 15, 28.
necessario, adv., necessarily, 16, 13.
necesse, adj., necessary, 18, 10.
negatio, f., business, 4, 9.
negligo, v., neglect, nullify, 19, 1.
negotiator, m., tradesman, 48, 1.
nempe, adv., for indeed, since, 14, 16.
nequeo, v., not to be able, 15, 19.
nimius, adj., excessive, 10, 7.
nisi, adv., if not, unless, 6, 8.
noto, v., to signify, make known, note, 8, 17.
numeratio, f., count out, consideration, 3, 1.
numerus, m., number, notation, 3, 6.
nunc, adv., now, 1, 10.
obiter, adv., immediately, in passing along, 1, 7.
observo, v., apply, 16, 6.
perceive, 6, 10.
occuro, v., to happen, 29, 28.
octo, num. adj., eight, 45, 28.
offero, v., present, produce, show, 9, 27.
olim, adv., originally, 1, 9.
omnis, adj., all, every, 4, 17.
operatio, f., operation, 51, 5.
operor, v., work, operate, 31, 18.
oppositus, adj., opposite, 6, 28.
opus, n., work, 49, 19.
ordinario, ordinary, 1, 3.
ordo, m., rows, 4, 15.
pars, f., part, 26, 1.
particula, f., small part, fractional part, 20, 7.
pateo, v., be clear, follow, 24, 9.
pecunia, f., money, 48, 3.
perago, v., complete, 4, 20.
peregre, adv., abroad, otherwise, 54, 18.
perficio, v., complete, make, 17, 19.
perspicio, v., understand, see, 4, 11.
pertineo, v., concern, relate to, 7, 18.
perturbatus, adj., mixed, 61, 13.
peruenio, v., arrive at, come up to, 58, 11.
phalanx, f., phalanx, folio, 45, 9.
plane, adv., readily, 4, 8.
pleo, v., fill, fulfill, complete, 16, 5.
plerumque, adv., frequently, 24, 7.
plico, v., to make manifold, 40, 4.
plurimus, adj., very much, 10, 15.
plus (compar. multus), adj., much more than, 10, 15.
pondus, neut., weight, 4, 9.
pono, v., place, put, propose, 6, 1.
porro. adv., then, now. 8, 8.
positio, f., position, situation, 44, 20.
possum, v., be able, possible, 1, 11.
posterior, comp. of posterus, adj., latter, following, 22, 12.
posthac, adv., hereafter, henceforth, 7, 9.
praecipus, adj., important, 39, 16.
praegnans, adj., pregnant, 41, 28.
praemissum, v., and adj., sent in advance, above, 4, 6.
praescriptus, adj., prefaced, adjoining, 7, 29.
primum, adv., first, 1, 11.
primus, adj., first, 2, 3.
prior, adj. (comp.), former, first, 31, 13.
priviligium, n., privilege, 1, 11.
probatio, f., trial, test, proof, 53, 8.
procreo, v., create, produce, 7, 19.
produco, v., produce, give, 3, 13.
productum, neut., product, 7, 10.
profero, v., bring forward, produce, give, 19, 16.
professor, m., professor, 47, 6.
progressio, f., progression, 47, 6.
prolongri, v., to extend, 8, 3.
propono, v., propose, set forth, 2, 3.

proporcio, f., connection, ratio, 14, 13.
proposicio, f., proposition, 51, 21.
propositus, part., proposed, 16, 23.
proprius, adj., particular, 3, 3.
prorsus, adv., entirely, 5, 17.
prosero, v., extend, 24, 18.
protraho, v., draw, extend, 8, 14.
prouenio, v., come forth, appear, 17, 24.
prout, adv., likewise, as, 11, 14.
punctum, neut., point, 17, 8.
quadruplus, adj., fourfold, 68, 4.
quadratum, neut., square, quadratic, 16, 29.
quadratus, adj., square, 16, 27.
quadruplatio, quadruplication, 18, 12.
quaesito, v., obtain, acquire, 22, 4.
qualis, adj., whatever, 7, 15.
quantitas, f., amount, quantity, 3, 14, exponent, 16, 7.
quantum, adv., as much as, 1, 11.
quartus, adj., fourth, 16, 9.
quasi, adv., as if, 41, 28.
quemadmodum, adv., just as, in the same way, 13, 2.
queo, v., can, be able, 20, 13.
quero, v., inquire, ask, 56, 28.
questio, f., question, problem, 7, 16.
quincuplam, adj., five fourths, 66, 25.
quintus, adj., fifth, 30, 19.
quomodo, adv., how, 3, 24.
quotquot, adj., however many, 17, 17.
radix, f., root, 3, 10.
raro, adv., seldom, 10, 13.
ratio, f., computation, 3, 5.
rationalis, adj., rational, 16, 4.
rectus, adj., straight, 17, 19.
reduco, v., bring back, reduce, 20, 21.
regula, f., rule, 2, 3.
relinquo, v., leave, 45, 19.
reperio, v., find again, understand, 24, 20; represent, 6, 26.
repeto, v., repeat, 28, 14.
represento, v., represent, 8, 21.
requiro, v., to be required, 11, 15.
res, f., root, unknown, 3, 10.
residuus, neut., remaining, 5, 32.
respectus, m., consideration, 6, 3.
respondeo, v., respond, answer, 45, 12.
satisfacio, v., satisfy, to be sufficient, 27, 26.
scio, v., know, understand, 7, 19.
scribo, v., write, 4, 13.
secundum, prep., according to, following, 5, 4.
secundus, adj., second, following, 2, 5.
semper, adv., always, 8, 21.
separaro, v., separate, 37, 23.
series, f., series, 55, 3.
sequor, v., follow, 3, 22.
seu, conj., or, 2, 4.
sexquitertius, adj., ratio of 4 to 3, 45, 24.
significans, p. part. (significo), significant, 39, 24.
significo, v., denoted, 3, 7.
signum, neut., sign, 3, 4.
similiter, adv., similarly, 4, 1.
simul, adv., at the same time, together, also, 19, 26.
singularis, adj., one by one, each, 35, 1.
societas, f., copartnership, 48, 1.
soleo, v., to use, be accustomed, 3, 7.
solidum, n., gold coin, 44, 25.
solucio, f., solution, 42, 18.
sortio, v., assign, result, 7, 13.
species, f., kind, species, 28, 23.
subducto, v., draw up, 5, 28.
subiectus, adj., accompanying, adjacent, 7, 20.
subscribo, v., write underneath, 5, 5.
subsequialterus, adj., one and one-half, 43, 28.
substantio, f., quantity, 3, 26.
subtractio, f., subtraction, 5, 26.
subtrahendo, withdrawing, 5, 27.
subtraho, v., subtract, 22, 25.
sugo, v., reduce, imbibe, 31, 6.
summa, f., amount, sum, 19, 15.
sumo, v., assume, take, 30, 8.
sumptio, f., assumption, 33, 1.
sumptus, m., value, 67, 6.
superius, comp., adv., above, 5, 2.
surdus, m., surd, irrational, 15, 15.
syllaba, f., syllable, 16, 22.
tabula, f., table, 7, 21.

tantum, adv., with quantum, as much, or the (very), 3, 27.
tempus, neut., time, 48, 14.
tertius, adj., the third, 7, 2.
toties, adv., as many times, 17, 17.
tractatus, m., treatise, 2, 2.
trado, v., impart, set forth, 14, 20.
triplex, adj., triple, 18, 13.
triplico, v., triple, 56, 24.
tunc, adv., then, immediately, 6, 26.
ultimus, adj., last, 18, 3.
undecimus, adj., the eleventh, 30, 23.
unitas, f., unity, unit, 17, 18.
unus, num., adj., one, 33, 13.
usus, m., use, 2, 8.
utilibus, adj., useful, 1, 13.
valeo, v., to be of value, signify, 14, 23.
valor, m., value, 39, 22.
varius, adj., varying, manifold, 16, 7.
venio, v., come, arrive, 10, 20.
vero, adv., truly, certainly, 5, 14.
vito, v., avoid, escape, 24, 18.
voco, v., call, designate, 16, 18.
volens, adj., inclined, desiring, 10, 19.
vulgaris, adj., common, 10, 29.

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